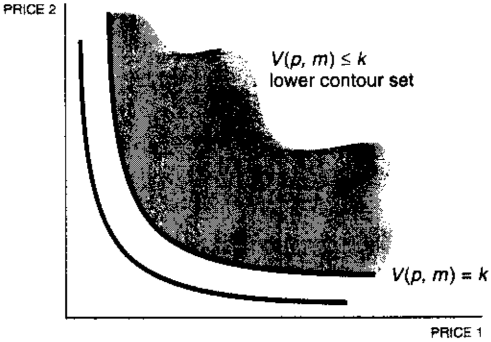

Figure 7.2 **Price indifference curves.** The indifference curve is all those prices such that $v(\mathbf{p}, m)) = k$, for some constant k. The lower contour set consists of all prices such that $v(\mathbf{p}, m) \leq k$.

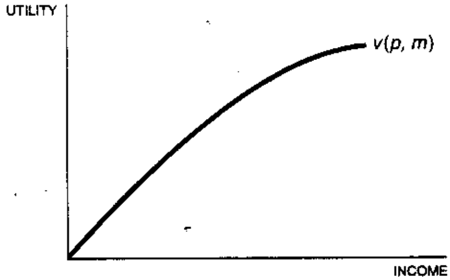

Figure 7.3 **Utility as a function of income.** As income increase indirect utility must increase.

An equivalent definition of the expenditure function is given by the following problem:

$$e(\mathbf{p}, u) = \min \ \mathbf{px}$$
$$\text{such that } u(\mathbf{x}) \geq u.$$

The expenditure function gives the minimum cost of achieving a fixed level of utility.

The expenditure function is completely analogous to the cost function we considered in studying firm behavior. It therefore has all the properties we derived in Chapter 5, page 71. These properties are repeated here for convenience.

Properties of the expenditure function.

(1) $e(\mathbf{p}, u)$ *is nondecreasing in* $\mathbf{p}$.

(2) $e(\mathbf{p}, u)$ *is homogeneous of degree* 1 *in* $\mathbf{p}$.

(3) $e(\mathbf{p}, u)$ *is concave in* $\mathbf{p}$.

(4) $e(\mathbf{p}, u)$ *is continuous in* $\mathbf{p}$, *for* $\mathbf{p} \gg 0$.

(5) If $\mathbf{h}(\mathbf{p}, u)$ *is the expenditure-minimizing bundle necessary to achieve utility level* u *at prices* $\mathbf{p}$, *then* $h_i(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial p_i}$ *for* $i = 1, \ldots, n$ *assuming the derivative exists and that* $p_i > 0$.

Proof. These are exactly the same properties that the cost function exhibits. See in Chapter 5, page 71 for the arguments. ∎

The function $\mathbf{h}(\mathbf{p}, u)$ is called the **Hicksian demand function**. The Hicksian demand function is analogous to the conditional factor demand functions examined earlier. The Hicksian demand function tells us what consumption bundle achieves a target level of utility and minimizes total expenditure.

A Hicksian demand function is sometimes called a **compensated demand function**. This terminology comes from viewing the demand function as being constructed by varying prices *and income* so as to keep the consumer at a fixed level of utility. Thus, the income changes are arranged to "compensate" for the price changes.

Hicksian demand functions are not directly observable since they depend on utility, which is not directly observable. Demand functions expressed as a function of prices and income are observable; when we want to emphasize the difference between the Hicksian demand function and the usual demand function, we will refer to the latter as the **Marshallian demand function**, $\mathbf{x}(\mathbf{p}, m)$. The Marshallian demand function is just the ordinary market demand function we have been discussing all along.

7.4 Some important identities

There are some important identities that tie together the expenditure function, the indirect utility function, the Marshallian demand function, and the Hicksian demand function.

Let us consider the utility maximization problem

$$v(\mathbf{p}, m^*) = \max u(\mathbf{x})$$
$$\text{such that } \mathbf{px} \leq m^*.$$

Let $\mathbf{x}^*$ be the solution to this problem and let $u^* = u(\mathbf{x}^*)$. Consider the expenditure minimization problem

$$e(\mathbf{p}, u^*) = \min \mathbf{px}$$
$$\text{such that } u(x) \geq u^*.$$

An inspection of Figure 7.4 should convince you that in nonperverse cases the answers to these two problems should be the same $\mathbf{x}^*$. (A more rigorous argument is given in the appendix to this chapter.) This simple observation leads to four important identities:

(1) $e(\mathbf{p}, v(\mathbf{p}, m)) \equiv m$. The minimum expenditure necessary to reach utility $v(\mathbf{p}, m)$ is m.

(2) $v(\mathbf{p}, e(\mathbf{p}, u)) \equiv u$. The maximum utility from income $e(\mathbf{p}, u)$ is u.

(3) $x_i(\mathbf{p}, m) \equiv h_i(\mathbf{p}, v(\mathbf{p}, m))$. The Marshallian demand at income m is the same as the Hicksian demand at utility $v(\mathbf{p}, m)$.

(4) $h_i(\mathbf{p}, u) \equiv x_i(\mathbf{p}, e(\mathbf{p}, u))$. The Hicksian demand at utility u is the same as the Marshallian demand at income $e(\mathbf{p}, u)$.

This last identity is perhaps the most important since it ties together the "observable" Marshallian demand function with the "unobservable" Hicksian demand function. Identity (4) shows the Hicksian demand function—the solution to the expenditure minimization problem—is equal to the Marshallian demand function at an appropriate level of income—namely, the minimum income necessary at the given prices to achieve the desired level of utility. Thus, any demanded bundle can be expressed *either* as the solution to the utility maximization problem or the expenditure minimization problem. In the appendix to this chapter we give the exact conditions under which this equivalence holds. For now, we simply explore the consequences of this duality.

It is this link that gives rise to the term "compensated demand function." The Hicksian demand function is simply the Marshallian demand functions for the various goods if the consumer's income is "compensated" so as to achieve some target level of utility.

A nice application of one of these identities is given in the next proposition:

Roy's identity. *If $\mathbf{x}(\mathbf{p}, m)$ is the Marshallian demand function, then*

$$x_i(\mathbf{p}, m) = -\frac{\dfrac{\partial v(\mathbf{p}, m)}{\partial p_i}}{\dfrac{\partial v(\mathbf{p}, m)}{\partial m}} \quad \textit{for} \quad i = 1, \cdots n$$

provided, of course, that the right-hand side is well defined and that $p_i > 0$ and $m > 0$.

Proof. Suppose that $\mathbf{x}^*$ yields a maximal utility of u^* at $(\mathbf{p}^*, m^*)$. We know from our identities that

$$\mathbf{x}(\mathbf{p}^*, m^*) \equiv \mathbf{h}(\mathbf{p}^*, u^*). \tag{7.2}$$