

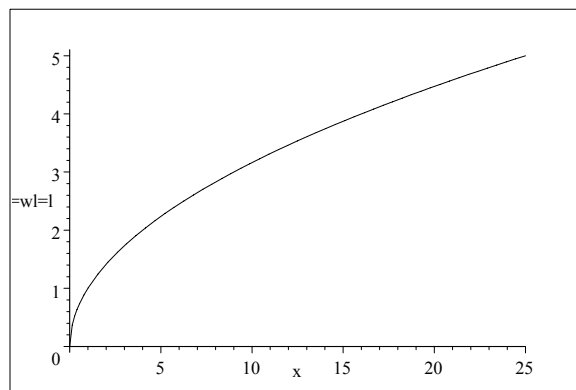
Deriving the production function from the cost function

March 20, 2001

The intent of these notes is to demonstrate that the production function can be derived from cost function.

Let's start with a simple graphical example, where production is a function of only one input, l , with a price of w . In this case, the cost function is $c = c(x, w)$. Graph this with c on the vertical axis and x on the horizontal axis. Hold w constant. To make life simpler assume $w = 1$. The function might look as follows.

$l^{.5}$

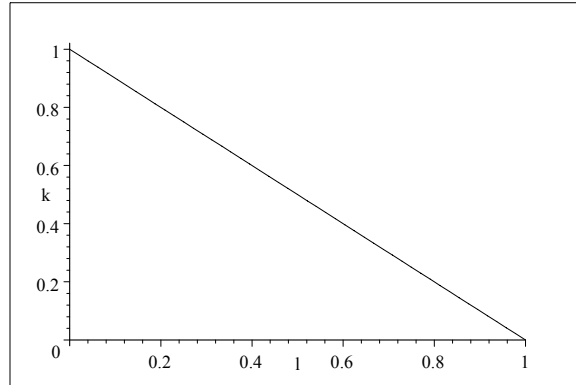


Now flip and rotate the page, looking at the graph with x on the vertical axis and l on the horizontal axis. What are you looking at?

Now assume $c = c(x, w, r)$, that is, assume two inputs.

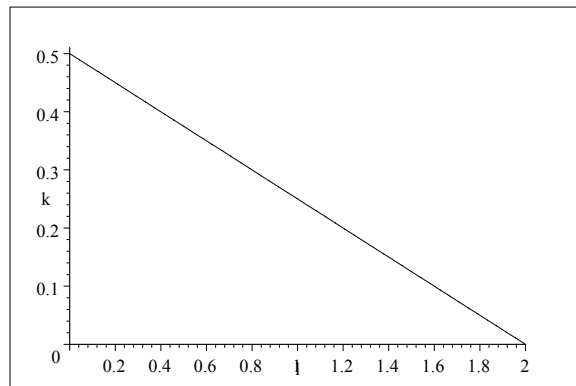
Choose an output level, e.g. $x = 1$ and see if you can use the cost function to derive the isoquant for one unit of output. If you can do this, you can do it for any output level, demonstrating that the production function can be derived from the cost function.

I will do this with the specific cost function $c = xw^{.5}r^{.5}$; that is $c = w^{.5}r^{.5}$ given that $x = 1$. Derive minimum cost when $w = r = 1$. It is 1. The corresponding isocost line is $k = 1 - l$. The general formula is $k = \frac{c}{r} - \frac{w}{r}l$



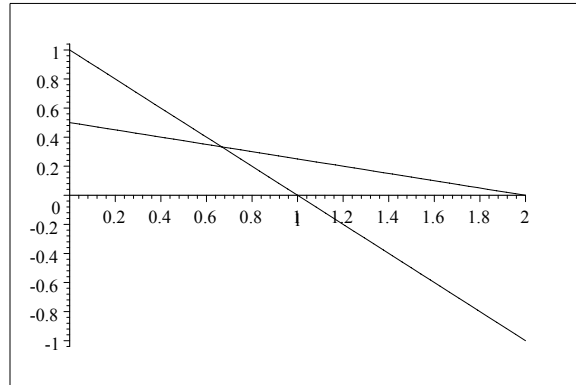
What do we know about the relationship between this isocost line and the isoquant for producing one unit of output. The isoquant cannot lie to the left of this line and must touch it at at least one point.

Now choose some other input prices. E.g. $w = 1$ and $r = 4$. With these input prices, the minimum cost of producing one unit of output is 2 and the corresponding budget line is $k = \frac{2}{4} - \frac{1}{4}l$



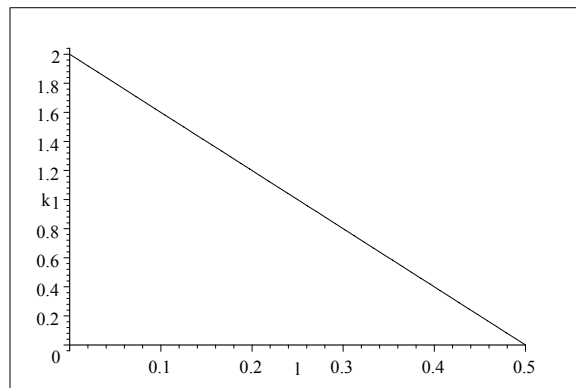
What do we know about the relationship between this isocost line and the isoquant for producing one unit of output. The isoquant cannot lie to the left of this line and must touch it at at least one point.

Putting these two isocost lines for one unit of output together we get

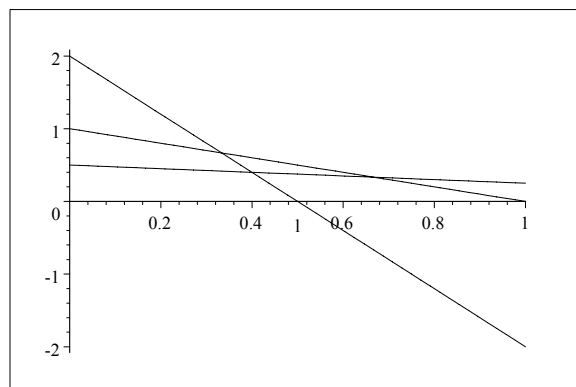


Note that the isoquant for $x = 1$ cannot lie to the left of either of these lines and must touch both at at least one point.

Now choose some other input prices. E.g. $w = 4$ and $r = 1$. With these input prices, the minimum cost of producing one unit of output is 2 and the corresponding budget line is $k = \frac{2}{1} - \frac{4}{1}l$



Again the isoquant for $x = 1$ cannot lie to the left of this line and must touch it at at least one point. Each of these isocost lines for $x = 1$ is providing additional information about the shape of the isoquant. Putting the three isocost lines together, one gets



We are tracing out the isoquant for $x = 1$. Just keep doing this for different combinations of the two input prices and the isoquant will get traced out.

So, we have derived the isoquant for $x = 1$ from knowledge of the cost function. We could have just as easily derived the isoquant for any level of x .

What have we demonstrated

$$c(x, w, r) \Rightarrow x = f(k, l)$$

Earlier we demonstrated that

$$c(x, w, r) \Leftarrow x = f(k, l)$$

Putting these together

$$c(x, w, r) \Leftrightarrow x = f(k, l)$$

Now let's do the same thing algebraically

Assume $c = xw^{.5}r^{.5}$ and find the corresponding $x = f(k, l)$

By Shepard's lemma

$$l_c = x \left(\frac{r}{w} \right)^{.5}$$

and

$$k_c = x \left(\frac{w}{r} \right)^{.5}$$

Solve each for $\left(\frac{w}{r}\right)^{.5}$ to get $\left(\frac{w}{r}\right)^{.5} = \frac{x}{l}$ and $\left(\frac{w}{r}\right)^{.5} = \frac{k}{x}$. That is $\frac{x}{l} = \frac{k}{x}$, Solution is: $\{x = \sqrt{(lk)}\}$, $\{x = -\sqrt{(lk)}\}$. That is $x = k^{.5}l^{.5}$.

More generally, assume $c = c(x, w, r)$. Derive the conditional input demand function $l = l(x, w, r)$ and $k = k(x, w, r)$. Now normalize prices by setting $r = 1$ to get $l = l(x, w, 1)$. Solve this for $w = w(x, l)$. Substitute this into $k = k(x, w, 1)$ to get $k = k(x, w(x, l), 1)$. Note that the remaining variables are x, l , and k . Solve for x to get the production function.