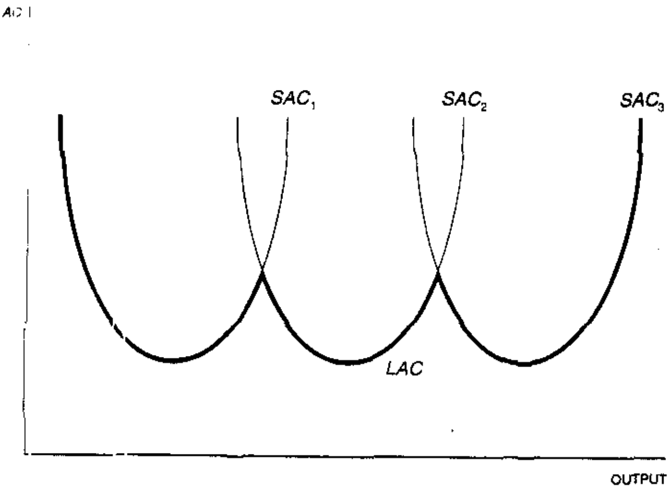

Long-run average cost curve. The long-run average cost curve, LAC, is the lower envelope of the short-run average cost curves, SAC_1, SAC_2, and SAC_3.

Properties of the cost function.

1) Nondecreasing in $\mathbf{w}$. *If* $\mathbf{w}' \geq \mathbf{w}$, *then* $c(\mathbf{w}', y) \geq c(\mathbf{w}, y)$.

2) Homogeneous of degree 1 in $\mathbf{w}$. $c(t\mathbf{w}, y) = tc(\mathbf{w}, y)$ *for* $t > 0$

3) Concave in $\mathbf{w}$. $c(t\mathbf{w} + (1-t)\mathbf{w}', y) \geq tc(\mathbf{w}, y) + (1-t)c(\mathbf{w}', y)$ *for* $0 \leq t \leq 1$.

4) Continuous in $\mathbf{w}$. $c(\mathbf{w}, y)$ *is continuous as a function of* $\mathbf{w}$, *for* $\mathbf{w} \gg \mathbf{0}$.

Proof.
1) This is obvious, but a formal proof may be instructive. Let $\mathbf{x}$ and $\mathbf{x}'$ be cost-minimizing bundles associated with $\mathbf{w}$ and $\mathbf{w}'$. Then $\mathbf{wx} \leq \mathbf{wx}'$ by minimization and $\mathbf{wx}' \leq \mathbf{w}'\mathbf{x}'$ since $\mathbf{w} \leq \mathbf{w}'$. Putting these inequalities together gives $\mathbf{wx} \leq \mathbf{w}'\mathbf{x}'$ as required.

2) We show that if $\mathbf{x}$ is the cost-minimizing bundle at prices $\mathbf{w}$, then $\mathbf{x}$ also minimizes costs at prices $t\mathbf{w}$. Suppose not, and let $\mathbf{x}'$ be a cost-minimizing bundle at $t\mathbf{w}$ so that $t\mathbf{wx}' < t\mathbf{wx}$. But this inequality implies $\mathbf{wx}' < \mathbf{wx}$, which contradicts the definition of $\mathbf{x}$. Hence, multiplying factor prices by a positive scalar t does not change the composition of a cost-minimizing bundle, and, thus, costs must rise by exactly a factor of t: $c(t\mathbf{w}, y) = t\mathbf{wx} = tc(\mathbf{w}, y)$.

3) Let $(\mathbf{w}, \mathbf{x})$ and $(\mathbf{w}', \mathbf{x}')$ be two cost-minimizing price-factor combinations

The same graph can be used to discover a very useful way to find an expression for the condition factor demand. We first state the result formally:

Shephard's lemma. *(The derivative property) Let $x_i(\mathbf{w}, y)$ be the firm's conditional factor demand for input i. Then if the cost function is differentiable at $(\mathbf{w}, y)$, and $w_i > 0$ for $i = 1, \ldots, n$ then*

$$x_i(\mathbf{w}, y) = \frac{\partial c(\mathbf{w}, y)}{\partial w_i} \qquad i = 1, \cdots, n.$$

Proof. The proof is very similar to the proof of Hotelling's lemma. Let $\mathbf{x}^*$ be a cost-minimizing bundle that produces y at prices $\mathbf{w}^*$. Then define the function

$$g(\mathbf{w}) = c(\mathbf{w}, y) - \mathbf{w}\mathbf{x}^*.$$

Since $c(\mathbf{w}, y)$ is the cheapest way to produce y, this function is always nonpositive. At $\mathbf{w} = \mathbf{w}^*$, $g(\mathbf{w}^*) = 0$. Since this is a maximum value of $g(\mathbf{w})$, its derivative must vanish:

$$\frac{\partial g(\mathbf{w}^*)}{\partial w_i} = \frac{\partial c(\mathbf{w}^*, y)}{\partial w_i} - x_i^* = 0 \qquad i = 1 \cdots, n$$

Hence, the cost-minimizing input vector is just given by the vector of derivatives of the cost function with respect to the prices. ∎

Since this proposition is important, we will suggest *four* different ways of proving it. First, the cost function is by definition equal to $c(\mathbf{w}, y) \equiv \mathbf{w}\mathbf{x}(w, y)$. Differentiating this expression with respect to w_i and using the first-order conditions give us the result. (Hint: $\mathbf{x}(\mathbf{w}, y)$ also satisfies the identity $f(\mathbf{x}(\mathbf{w}, y)) \equiv y$. You will need to differentiate this with respect to w_i.)

Second, the above calculations are really just repeating the derivation of the envelope theorem described in the next section. This theorem can be applied directly to give the desired result.

Third, there is a nice geometrical argument that uses the same Figure 5.4 we used in arguing for concavity of the cost function. Recall in Figure 5.4 that the line $c = w_1 x_1^* + \sum_{i=2}^{n} w_i x_i^*$ lay above $c = c(\mathbf{w}, y)$ and both curves coincided at $w_1 = w_1^*$. Thus, the curves must be tangent, so that $x_1^* = \partial c(\mathbf{w}^*, y)/\partial w_1$.

Finally, we consider the basic economic intuition behind the proposition. If we are operating at a cost-minimizing point and the price w_1 increases, there will be a direct effect, in that the expenditure on the first factor will increase. There will also be an indirect effect, in that we will want to change the factor mix. But since we are operating at a cost-minimizing point, any such infinitesimal change must yield zero additional profits.

(3) $e(\mathbf{p}, u)$ *is concave in* $\mathbf{p}$.

(4) $e(\mathbf{p}, u)$ *is continuous in* $\mathbf{p}$, *for* $\mathbf{p} \gg 0$.

(5) If $\mathbf{h}(\mathbf{p}, u)$ *is the expenditure-minimizing bundle necessary to achieve utility level* u *at prices* $\mathbf{p}$, *then* $h_i(\mathbf{p}, u) = \dfrac{\partial e(\mathbf{p}, u)}{\partial p_i}$ *for* $i = 1, \ldots, n$ *assuming the derivative exists and that* $p_i > 0$.

Proof. These are exactly the same properties that the cost function exhibits. See in Chapter 5, page 71 for the arguments. ∎

The function $\mathbf{h}(\mathbf{p}, u)$ is called the **Hicksian demand function**. The Hicksian demand function is analogous to the conditional factor demand functions examined earlier. The Hicksian demand function tells us what consumption bundle achieves a target level of utility and minimizes total expenditure.

A Hicksian demand function is sometimes called a **compensated demand function**. This terminology comes from viewing the demand function as being constructed by varying prices *and income* so as to keep the consumer at a fixed level of utility. Thus, the income changes are arranged to "compensate" for the price changes.

Hicksian demand functions are not directly observable since they depend on utility, which is not directly observable. Demand functions expressed as a function of prices and income are observable; when we want to emphasize the difference between the Hicksian demand function and the usual demand function, we will refer to the latter as the **Marshallian demand function**, $\mathbf{x}(\mathbf{p}, m)$. The Marshallian demand function is just the ordinary market demand function we have been discussing all along.

7.4 Some important identities

There are some important identities that tie together the expenditure function, the indirect utility function, the Marshallian demand function, and the Hicksian demand function.

Let us consider the utility maximization problem

$$v(\mathbf{p}, m^*) = \max u(\mathbf{x})$$
$$\text{such that } \mathbf{p}\mathbf{x} \leq m^*.$$

Let $\mathbf{x}^*$ be the solution to this problem and let $u^* = u(\mathbf{x}^*)$. Consider the expenditure minimization problem

$$e(\mathbf{p}, u^*) = \min \mathbf{p}\mathbf{x}$$
$$\text{such that } u(x) \geq u^*.$$