Econ 6808 Introduction to Quantitative Analysis

review questions -set 2.

III. Economic Curvature and Economic Applications of Duality Theory

Review questions.

1. What does it mean to say that the function $y = g(m_1, m_2)$ is *concave*? Give an answer in both words and in mathematical notation.

2. Convince me that the cost function, $c = c(y, \mathbf{w})$ where $\mathbf{w} = [w_1, w_2, ...w_N)$, is not strictly convex in input prices. **Assume** that this cost function is increasing in y, increasing in **w**, and everywhere twice differentiable. As part of your answer define *strictly convex*.

answer: Consider the cost function c(y,w) where y is scaler output and w is the vector of input prices. This function is strictly convex in w if $\forall$ w^0 and w^1

$$c(y, \lambda w^0 + (1-\lambda) w^1) <> \lambda c(y, w^0) + (1-\lambda) c(y, w^1) \quad 1<\lambda<1.$$

Using the definition of the cost function, one can prove that the cost function must be homogenous of degree one in w. Homogenous of degree one in w is inconsistent with the cost function being strictly convex in w. Homogenous of degree one in w says that if input prices are increased holding relative input prices constant, costs will increase at a linear rate. If costs are increasing at linear rate the function is not strictly convex along these rays, and if it is not strictly convex along some linear rays in w space, the function is not strictly convex.

Proof that the cost function is homogenous of degree one in w:

> If x is the input vector that minimizes the cost of producing y given w, then w'x = c(y, w). If all input prices are multiplied by λ, $\lambda > 1$, relative input prices remain the same, so the cost minimizing input vector remains x. Therefore the minimum cost of producing y given input prices λw is (λw)'x = c(y, λw) = λ(w'x) = λc(y, w), and this is the definition of homogeneity of degree one in w for the function c(y, w).

Alternatively, one could demonstrate that the cost function must be concave in w, so can't be strictly convex in w.

> Proof that c(y, w) is concave in w: Define three input price vectors, w^0, w^1, w^{11} where w^{11} is a linear combination of w^0 and w^1 ($w^{11} = \lambda w^0 + (1-\lambda) w^1$).

Let the associated cost minimizing input vectors be x^0, x^1, and x^{11}. Therefore

$$c(y, w^{11}) = w^{11}x^{11} = (\lambda w^0 + (1-\lambda)w^1)x^{11} = \lambda w^0 x^{11} + (1-\lambda)w^1 x^{11} \geq \lambda c(y, w^0) + (1-\lambda)c(y,w^1)$$

because $w^0 x^{11} \geq c(y, w^0)$ (while x^{11} will produce y it is not the cost minimizing input vector given w^0), and for the same reason $w^1 x^{11} \geq c(y, w^{10})$. Therefore

$$c(y, w^{11}) = w^{11}x^{11} = (\lambda w^0 + (1-\lambda)w^1)x^{11} = \lambda w^0 x^{11} + (1-\lambda)w^1 x^{11} \geq \lambda c(y, w^0) + (1-\lambda)c(y,w^1)$$

proving that the cost function is concave in w.

Intuitively, think about what happens if the price of an input increases, holding other prices and output constant. If the firm continues to use the same input vector that was cost minimizing before the price increase, costs will increase at a linear rate as a function of the input price. Note that a linear rate is consistent with the cost function being concave in input prices but not consistent with it being strictly convex in input prices. However, often the production manager will be able to better than continuing to use the same input vector after an input price increases, in which case costs will increase at less than a linear rate. That is be strictly concave in input prices. The same argument hold when multiple prices change. When input prices increase, minimum costs will never increase at more than a linear rate.

3. Consider a cost function, c = c(y, **w**) where **w** = $[w_1, w_2, \ldots w_N)$. **Assume** that this cost function is nondecreasing in y, nondecreasing in **w**, quasiconcave in **w**, and everywhere twice differentiable. What does it mean to say that this cost function is *nondecreasing in y*? Use the definition of a cost function to prove (to convince me) that a cost function must by nondecreasing in y.

4. Consider a cost function, c = c(y, **w**) where **w** = $[w_1, w_2, \ldots w_N)$. **Assume** that this cost function is nondecreasing in y, nondecreasing in **w**, quasiconcave in **w**, and everywhere twice differentiable. What does it mean to say that this cost function is *nondecreasing in **w***? Use the definition of a cost function to prove that a cost function must by nondecreasing in **w**.

5. Define *Shephard's Lemma* in terms of production theory. Prove it.

6. Define both in words, and in functional notation the competitive firm's cost function and its conditional input demand functions.

answer: The cost function c(y, w) identifies the minimum cost of producing y units of output given the vector of input prices w. The firm's conditional input demand function identify the cost minimizing quantity of input j, j=1,2,...J, to produce y units of output given input prices w. The conditional input demand functions are of the form

$x_j^c = x_j^c(y, w)$

7. Convince me that an individual, Wilma, who is maximizing her utility, u(x, y), subject to a budget constraint will behave **as if** her utility function is *quasiconcave* even if it is not. You might want to convince me using graphs of indifference curves and budget lines to identify utility maximizing bundles. As part of your answer, define quasiconcave for this utility function, and explain what it tells you about the shape of the indifference curves. Assume that Wilma's utility function is increasing in x and y. The point of this exercise is to demonstrate that it is not restrictive to assume that a utility function that is increasing in its arguments is also quasiconcave.

8. Assume that Wilma's utility function, u(x, y) is increasing in x and y. Given this, what is gained by assuming that it is *strictly quasiconcave* rather than just *quasiconcave*?

9. Consider an individual's utility function, u(**x**) where **x** = $[x_1, x_2,...x_N]$, that has all the standard properties (increasing in **x**, quasiconcave in **x**, and twice differentiable. One can think of this utility function as a production function where utility is being produced using goods as inputs.

   Given this, define in both words and functional notation the consumer theory analog of the competitive firm's *cost function*. This function is called the *expenditure function* where E is conventionally used as the dependent variable. Duality theory tells us that this expenditure function is dual to the *direct* utility function, u(**x**), and also describes the individual's preferences.

   Then define both in words and in functional notation the consumer theory analog of a firm's conditional demand functions for inputs. In consumer theory, these are called the *hicksian demand functions*.

   Can one use the expenditure function to derive these hicksian demand function? How?

   Assume a simple functional form for a two-good utility function and derive both the expenditure function and the hicksian demand functions.

10. Intuitively explain, using a graphical analysis, how one can derive the production function, y = f(k, l) from the cost function c = c(y, r, w).

11. Choose some explicit cost function, and from it derive the corresponding

production function.

12. Define an individual's compensating variation for a change from (m^0, p^0) to (m^1, p^1) in four ways: in words, in terms of an indifference relationship between two states of the world, in terms of the indirect utility function , and in terms of the expenditure function.

Answer: The individual's compensating variation for a change from (m^0, p^0) to (m^1, p^1) is the amount of money that has to be subtracted from m^1 to make him indifferent between (m^0, p^0) and $(m^1 - cv, p^1)$. That is $(m^0, p^0) \sim (m^1 - cv, p^1)$.

In terms of the indirect utility function, it is

$$V(m^0, p^0) = V(m^1 - cv, p^1)$$

and in terms of the expenditure function it is

$$cv = E(U^1, p^1) - E(U^0, p^1)$$

13. Explain how one can derive a competitive firms cost function, c = c(y, **w**) where **w** = $[w_1, w_2,...w_N)$ from its production function y = f(**x**) where **x** = $[x_1, x_2,...x_N]$. Now assume a simple functional form for a 2-input production function and derive the firm's cost function.

14. Demonstrate that the production function $X = L^{.5}K^{.5}$, is strictly quasi-concave in terms of L and K. Hint: think about the shape of the isoquants for this production function and whether this production function is increasing in its arguments. As part of your answer define quasi-concavity of the production function in terms of L and K

answer: The production function $f(K,L) = L^{.5}K^{.5}$ is quasiconcave in L and K iff all of the upper level sets

$$L_u(X) \equiv = [(L,K) : L^{.5}K^{.5} \geq X]$$

are strictly convex sets. These upper level sets are called input requirement sets. This production function is strictly increasing in K and L because its partial derivitives are strictly positive for all positive values of K and L, so the input requirement set for output X is the isoquant for x and all input combinations to the right of the isoquant. The strict convexity of the input requirement sets can be demonstrated by showing that the isoquants are all negatively sloped such that the slope of the isoquant becomes flatter as L increases if L is on the horizontal axis.

The slope of the isoquant, $\frac{\partial K}{\partial L} \mid dX=0 =$, can be derived by taking the total differential of the production

function, which is equal to zero along the isoquant. That is,

$dX = \frac{\partial XU}{\partial L} dL + \frac{\partial XU}{\partial K} dK = 0$ solving for dK/dL one obtains

$\frac{dK}{dL} = - \frac{\partial X/\partial L}{\partial X/\partial K} = - \frac{.5L^{-.5}K^{.5}}{.5L^{.5}K^{-.5}} = - \frac{K}{L} < 0$, demonstrating that the slope of the isoquants are all

negative.

Now note that

$$\frac{(\partial \frac{dK}{dL})}{dL} = \frac{K}{L^2} > 0$$ which demonstrates that each negatively sloped isoquant becomes flatter (less negatively sloped) as L increases. Therefore, the isoquants for this production function have the nice shape we see in intermediate micro books and one can visually see from a graph of a representative isoquant that the upper level sets are strictly convex sets.