Economics 4999 Logic and Mathematics in Economics

Review questions on game theory

1. In a paragraph or so, define game theory to a friend who has never taken an economics course.

2. Prove that the Nash equilibrium is not always efficient. As part of your answer define efficient.

   answer: An allocation (payout) is efficient if there is no other allocation (payout) where at least one player is better off and neither player is worse off. How to prove NE are not always efficient? The easiest way is with a counter example. That is an example of a game where the NE is not efficient. For example Why? UL is a nash equilibrium but UL is not efficient because at DR both players are better off.

	left	right
up	$5, 6$	$8, 2$
down	$4, 9$	$6, 9$

3. Esther and Edgar met at a rave, but were, at the time, incapable of exchanging phone numbers. They desperately want to meet again. There is another rave tonight, and, if they both go, they will meet. However, they both have exams early tomorrow morning (6 a.m.). So, there would be some benefit to staying home and studying. The payout matrix is They think they might really like each other. They also want to pass their exams but passing up true love is hard to do. Are there Nash equilibruim in pure strategies? If so, identify them. Convince me that they will both end of at the rave.

		Esther	
		rave	study
Edgar	rave	$1000, 1000$	$0, 0$
	study	$0, 0$	$10, 10$

   answer: There are two Nash equilibrium in pure strategies. Both at rave or both home studying. Both at the rave is preferred to both home studying. Both understand the game, so figure out that the rational thing for the other player is to choose the rave.

4. Roger and Michelle, although they greatly enjoy each other's company, have very different preferences. Roger's tastes run to ladies' mud wrestling, while Michelle prefers Italian opera. They are planning what to do on Saturday night. Each has two options: go to the opera or go to mud wrestling. The payout matrix is They love each other dearly and will be greatly upset if they choose different venues.

		Michelle	
		wrestling	opera
Roger	wrestling	$2, 1$	$0, 0$
	opera	$0, 0$	$1, 2$

However, Roger prefers they end up at mud wrestling and Michelle prefers La Scala. Is this a zero-sum game? As part of your answer define zero-sum game. Does this game have a dominant strategy equilibrium? Find this game's two pure-strategy Nash equilibrium. If there is a mixed-strategy equilibrium, my guess is will be each player playing each option 50% of the time. Try to demonstrate that my intuition is correct. This game is called the battle of the sexes. Is one of these equilibrium more likely than the others? If so, which one and why.

answer: This is a not a zero-sum game (the two payouts in each box do not sum to zero). The two Nash equilibrium in pure strategies are everyone at mud wrestling or everyone at the opera. Therefore, in this game

$$\begin{aligned} EB_R &= p\pi_q 2 + p(1-\pi_q)(0) + (1-p)\pi_q(0) + (1-p)(1-\pi_q)1 \\ &= 2p\pi_q + (1-p)(1-\pi_q) \end{aligned}$$

and

$$\begin{aligned} EB_M &= q\pi_p 1 + q(1-\pi_p)0 + (1-q)\pi_p(0) + (1-q)(1-\pi_p)2 \\ &= q\pi_p + 2(1-q)(1-\pi_p) \end{aligned}$$

$\frac{d}{dp}\left[2p\pi_q + (1-p)(1-\pi_q)\right] = [3\pi_q - 1]$ and $\frac{d}{dq}\left[q\pi_p + 2(1-q)(1-\pi_p)\right] = [3\pi_p - 2]$

where p is the probability that Roger will choose mud wrestling, q is the probability that Michelle will choose mud wrestling. π_q is Roger's subjective probability that Michelle will choose mud wrestling, and π_p is Michelle's subjective probability that Roger will choose mud wrestling. Start with Roger's maximzation problem

$$\frac{\partial EB_R}{\partial p} = 3\pi_q - 1 \begin{matrix} > \\ = \\ < \end{matrix} 0 \text{ depending on the valueof } \pi_q$$

Note that the derivative of the Roger's objective function with respect to its choice variable p is not a function of p. So, Roger can't choose p to make this equal zero. There in no interior solution for Roger unless $\pi_q = .333$. However, if $\pi_q = .333$, the Roger won't care what p he chooses.

By symmetry, we know that Michelle won't care what q she chooses as long as $\pi_p = .666$. Demonstrating that the mixed-strategy equilibrium is $p = .666$ and $q = .333$. They each choose their favorite 2/3 of the time. What happened to my intuition. Interestingly the answer book says the mixed-strategy equilibrium is $p = q = .5$. If Michelle (or Roger) played .5 what would the other one want to do? The other one would want to play their favorite all of the time. One can see this by plugging .5 for π_q into EB_R.

$$EB_R = p.5(2) + .5\,(1-p) = .5 + .5p$$

This is continuously increasing in p, so Roger will want to set $p = 1$. What is interesting is that the mixed-strategy equilibrium kind of sucks. The probability that they will both end up at mud wrestling is $2/9 = (2/3)(1/3)$. The probability that Roger will end up at mud wrestling and Michelle at the opera is $4/9 = p(1-q) = (2/3)(2/3)$. The probability that Roger will end up at the opera and Michelle at mud wrestling is $1/9 = (1-p)q = (1/3)(1/3)$, and the probability that they will both end up at the opera is $2/9 = (1-p)(1-q) = (1/3)(2/3)$. 5/9 of the time they will end up apart. What is the expected benefits to both parties if they play the mixed-strategy Nash equilibrium.

$$EB_R = 2(2/3)(1/3) + (1/3)\,2/3) = 2/3$$

$$EB_M = (1/3)(2/3) + 2\,(2/3)\,(1/3) = 2/3$$

Compare this to the expected benefits they each would get if $p = q = .5$

$$EB_R = 2(.5)(.5) + (.5)\,(.5) = .75$$

$$EB_M = (.5)(.5) + 2\,(.5)\,(.5) = .75$$

Expectationally, they both do better with $p = q = .5$, but it is not a Nash equilibrium.

1. Consider the game **Dueling Saabs**. It is played by adolescent females who attend Boulder High School. They all own Saab convertibles with which they like to play chicken.[1] Every time two females encounter one another they are compelled to play chicken. They drive their Saabs at each other at breakneck speed and at the last second either swerve or not swerve. If one swerves she is a wimp. Those who do not swerve are considered aggressive. Ceteris paribus, aggressive is considered a desirable trait (it attracts adolescents boys), but aggressive can be costly if one encounters another aggressive driver. Assume the payout matrix for this game in numer of boys attracted or repelled is[2]

	swerve	don't swerve
swerve	$1, 1$	$0, 5$
don't swerve	$5, 0$	$-3, -3$

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[1] Care insurance is too costly for adolescent males so they remain car-less.

[2] The blood and guts resulting from accidents grosses out the boys.

While wandering through adolescence each female will encounter many other Saabs. Suppose that she cannot tell in advance whether the other Saab driving adolescent will act aggressively or wimp out. The payout to adopting either strategy depends on the proportion of the other females that act aggressively

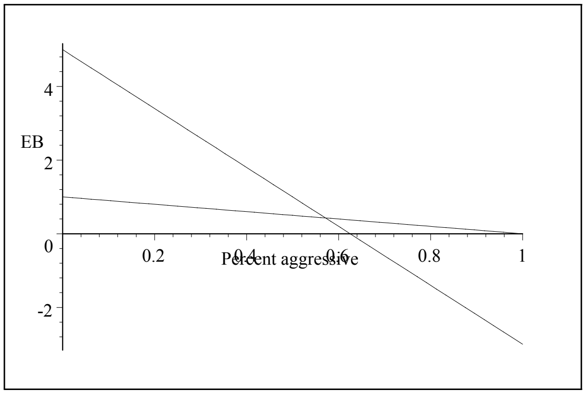
- Convince me that there cannot be an equilibrium in which all female Saab drivers are wimps.
- Convince me that there cannot be an equilibrium in which all female Saab drivers are aggressive.
- Since there is not an equilibrium where every female chooses the same strategy, determine whether there is an equilibrium where some proportion of the adolescent females choose to act aggressively, and the rest wimp out.
- If the proportion of aggressive drivers is greater than the equilibrium of aggressive drivers, which stragegy does better? If the proportion of aggressive drivers is less than the equilibrium of aggressive drivers, which stragegy does better?
- If the proportion of aggressive drivers is not the equilibrium proportion, will the proportion move back towards the equilibrium proportion as either wimps get more aggressive or aggressives become wimps.

1. **answer:**

- If everyone is a wimp, any individual driver can increase her expected benefits by becoming aggressive. (EB if all wimps $= 1$, your EB if your aggressive and everyone else is a wimp $= 5$). So, all wimps is not a Nash equilibrium. Given what everyone else is doing, you want to change your behavior.
- If everyone is aggressive, any individual driver can increase her expected benefits by becoming a wimp. (EB if all aggressive $= -3$, your EB if you are a wimp and everyone else is aggressive $= 0$). So, everyone aggressive is not a Nash equilibrium.
- $$EB_W = 1(1-\pi_a) + 0\pi_a = (1-\pi_a)$$

  and
  $$EB_A = 5(1-\pi_a) - 3\pi_a = 5 - 5\pi_a - 3\pi_a = 5 - 8\pi_a$$

  where π_a is the percent that are aggressive. In equilibrium, $EB_W = EB_A$, otherwise incentive to switch. Solving $(1-\pi_a) = 5 - 8\pi_a$, Solution is: $\left\{\pi_a = \frac{4}{7}\right\}$. Graphically, $1-\pi_a$



Equilibrium to the wimp game

If $\pi_a > \frac{4}{7}$ wimps do better. If $\pi_a < \frac{4}{7}$, the aggressive drivers do better.

- Yes
- As an aside, is DL or UR Nash equilibrium in pure strategies. Looks like it, but what would it mean? In a world of just two girls, one always wimping and the other not is a Nash equilibrium in pure strategies? What if there were hundred girls, and fifty were always wimps and 50 were always aggressive and everyone knew who was a wimp? This would not be a Nash equil. Because aggressive drivers would not want to be aggressive when they meet another aggressive. The same for wimps

1. Discuss game theory as a tool for modeling the behavior of ologopoly firms. As part of your answer, discuss the important factors one must consider in the modeling of the behavior of firms in a ologopolistic industry, and explain how game theory might be used to model those factors. As part of your answer, provide a game-theoretic model of ologopoly behavior. As part of your answer, discuss the notion of Nash equilibrium as it would apply in the context of equilibrium in an ologoplistic industry.

   answer: Game theory is designed to model the interactions among a small number of players where the actions of each affects all of the others. Ologopoly industries have a small number of firms where the actions of each firm in the industry affects all of the other firms in the industry. The payouts of the game are profits. Game theory assumes rational behavior on the part of the players, a reasonable starting assumption for ologopoly firms. Nash equilibrium in this context means

- each firm in the industry is doing what all of the other firms in the industry expected it to do.
- each firm is maximizing its profits given the behavior of the other firms in the industry

example: Joe Perrier is the sole owner of a spring that costlessly burbles forth as much mineral water as Joe cares to bottle. It costs Joe \$2/gallon to bottle the water (green bottles are expensive). The inverse demand function for Joe's water is

$$p = 20 - .2q$$

where $p \equiv$ price per gallon and $q \equiv$ number of gallons sold. Joe's profits as a function of q are

$$\begin{aligned} \pi(q) &= (20 - .2q)q - 2q \\ &= 18q - .2q^2 \end{aligned}$$

Start by finding the profit maximizing level of output for this monopolist. Look for critical points

$$\frac{\partial \pi(q)}{\partial q} = 18 - .4q$$

Set this equal to zero and solve for q $18 - .4q = 0$, Solution is: $\{q = 45.0\}$. Check the second-order conditions

$$\frac{\partial^2 \pi(q)}{\partial q^2} = -.4q < 0$$

So $q^* = 45$ and Joe's monopoly profits are

$$\pi(45) = 18(45) - .2(45)^2 = 405$$

Now suppose Joe's neighbor Wilbur Evian discovers a mineral spring on his property and no one can taste the difference between the two waters. It costs \$6/gallon to pump and bottle Evian. Total demand form mineral water remains as before

$$p = 20 - .2q$$

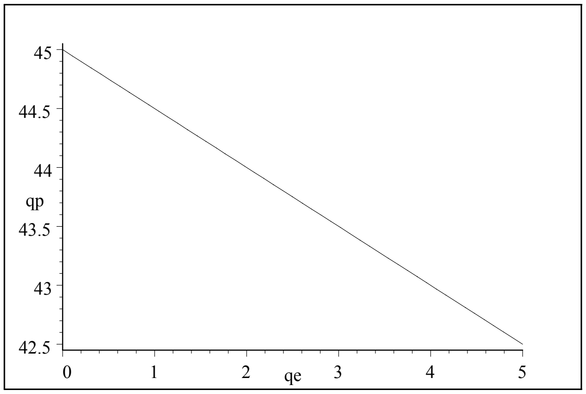
Both guys are dense, assuming that the other guy's production decision is independent of what they do (silly but true). Find the Nash equilibrium. Start by finding Perrier's supply function

$$\begin{aligned} \pi(q_p) &= [20 - .2(q_p + q_e)]\, q_p - 2q_p \\ &= 18q_p - .2q_p^2 - .2q_p q_e \end{aligned}$$

Perrier's profit maximizing level of output is found

$$\frac{\partial \pi_p}{\partial q_p} = 18 - .4q_p - .2q_e$$

Set this equal to zero and solve for q_p $18 - .4q_p - .2q_e$, Solution is: $\{q_p = 45.0 - 0.5q_e\}$. This is Joe Perrier's suppy function for Perrier as a function of q_e. Graphically, $45.0 - 0.5q_e$



Joe's reaction function

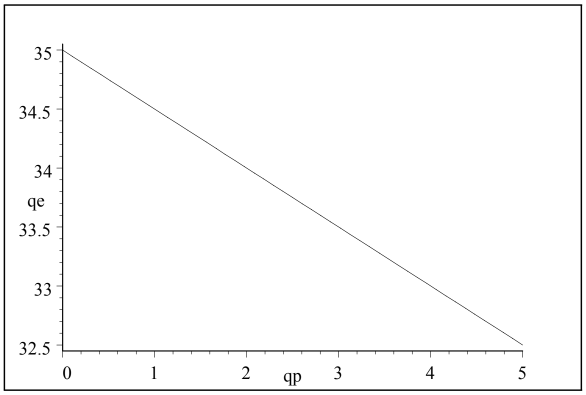
This supply function is Joe's eaction function in that it tells us how much Joe will choose to produce as a function of what Wilbur produces. Now find Wilbur Evian's supply function

$$\begin{aligned} \pi(q_e) &= [20 - .2(q_p + q_e)]\, q_e - 6q_e \\ &= 14q_e - .2q_e^2 - .2q_p q_e \end{aligned}$$

Perrier's profit maximizing level of output is found

$$\frac{\partial \pi_e}{\partial q_e} = 14 - .4q_e - .2q_p$$

Set this equal to zero and solve for q_e , $14 - .4q_e - .2q_p = 0$, Solution is: $\{q_e = 35.0 - 0.5q_p\}$ This is Wilburs suppy function for Perrier as a function of q_p. Graphically, $35.0 - 0.5q_p$



Wilbur's reaction function

The Nash equilibrium is the solution to the two reaction functions and $\begin{array}{c} q_e = 35.0 - .5q_p \\ q_p = 45.0 - 0.5q_e \end{array}$, Solution is: $\{q_e = 16.667, q_p = 36.667\}$. That is, if Wilbur bottles 16.667 Evians and Joe bottles 36.667 Perriers, neither will want to change their output level.

1. Assume a two-person, two-strategy, one-shot game. What characteristic's does the payout matrix have to have for this game to be a Prisoner's Dilemma. Specifically,

- List a condition that is necessary, but not sufficient
- List the conditions that are necessary and sufficient
- List conditions that are sufficient but not necessary

1. **answer:** A condition for PD that is neccesary is one whose absence ⇒ the game is not PD. A condition is sufficient is one whose existence ⇒ PD.The following are each examples of conditions that are necessary but not sufficient for PD

- Player 1 has a dominant strategy
- Player 2 has a dominant strategy
- Both players have dominant strategies

None of these conditions are sufficient because it is possible to write down a game where the condition holds but the game is not PD. Each of these conditions is necessary because it is impossible to write down a PD game where these three conditions do not hold. Second part: Conditions that are both necessary and sufficient for PD are the defintion of a PD game; that is

$$\text{(definition of PD)} \Leftrightarrow \text{(conditions that are necc and suff for PD)}$$

The necc and suff conditions are (1) there is a dominant stragegy equil and no other Nash equilibrium. and (2) If both players play their nondominant strategy (which is obviously not an equilibrium), both players get a payout that is $\geq$ his or her equilibrium payout. Now express things in terms of the payout matrix

$$\begin{array}{cccc} & & \multicolumn{2}{c}{2} \\ & & \text{left} & \text{right} \\ 1 & \text{up} & a, b & \boxed{c, d} \\ & \text{down} & e, f & \boxed{g, h} \end{array}$$

Remember the game is the same is left is switched for right and up is switched for down. Necessary and sufficient conditions for PD are (1)

$$\begin{aligned} a &> e \\ b &> d \\ c &\geq g \\ f &\geq h \end{aligned}$$

but either $c > g$, $f > h$ or both. And (2)

$$\begin{aligned} a &\leq g \\ b &\leq h \end{aligned}$$

but either $a < g$, $b < h$, or both. Why? Note that

$$(a > e \text{ and } c \geq g) \Rightarrow \text{ up is player's 1's dominant stragegy}$$

and

$$(b > d \text{ and } f \geq h) \Rightarrow \text{ left is player's 2's dominant stragegy}$$

So, UL is a dominant strategy equilibrium and UR and DL are not Nash equilibrium. If one adds the additional condition that either $c > g$, $f > h$, or both then DR is not a Nash equilibrium.[3] So (1)$\Rightarrow$ UL is a dominant strategy equilibrium and there are no other equilibrium. Now look at (2)

$$\begin{gathered} a \leq g \\ b \leq h \end{gathered}$$

but either $a < g$, $b < h$, or both.

$(a \leq g$ and $b \leq h) \Rightarrow$ (both players are at least as well off at DR as they are at UL)

and

(either $a < g$, $b < h$, or both) $\Rightarrow$ (at least one of the players is strictly better off at DR)

So, in summary, (1) and (2) together are necessary and sufficient for the game to be a PD. Now consider part 3 of the question. An example will suffice

		2	
		left	right
1	up	$5, 6,$	$8, 2$
	down	$4, 9$	$6, 9$

This game is PD so it is sufficient for PD but there are many PD games that do not have this payout matrix, so it is not necessary. This concludes my answer. Note that I have defined the PD game such that the weakly preferred outcome occurs when both players play their nondominant strategies. One could generalize the definition of the PD game such that their has to be an outcome that is preferred to the dominant strategy equilibrium but that is not necessarily the outcome with both player playing their nondominant strategy. Richard, in class, game up with an example where DL or UR was preferred to the dominant strategy equilibrium. In this case the necc and sufficient conditions outlined above are sufficient but not necessary.

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[3] Note that if $c = g$ and $f = h$ then DR is Nash equilibrium