1 Derivation of the cost function: The production manager's problem

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Let's consider the cost function in a bit more detail. Where does it come from?

Imagine that you have graduated from C.U. (maybe hard to imagine given that you are taking econ 4808) and you have been hired by a firm to run its widget factory; that is, you are in charge of production.

On your first day, the old guy that you are replacing describes to you the state of knowledge for producing widgets. That is, he explains that you will need labor and/or capital to produce widgets, and explains how many widgets can be produced at every possible nonnegative amount of labor and capital. That is, he tells you the production function.

Every morning when you get to work you read the NYTimes to determine the going rate for labor and capital, w and r. (You are a competitive buyer of these two inputs). Every afternoon you fire everyone and hire anew the next morning.

The minute you put down the paper, the marketing department calls to tell you how many widgets need to be produced that day.

Each day your goal is to hire the amount of labor and capital that minimizes the cost of producing whatever number of widgets you were told to produce. Or at least this is what we are going to assume is you goal.[1]

Your problem is the *production manager's problem.*

Your choice variables (the endogenous variables) are the amount of labor, l, and capital k to hire.

You are constrained by the number of widgets you must produce, x, input prices, w and r, and the state of technical knowledge for producing widgets. So, your exogenous variables are x, w, and r.

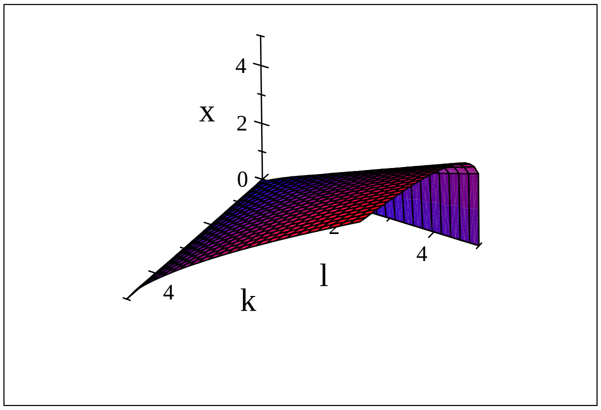
There are lots of ways to describe the state of technical knowledge for producing widgets, one of which is the production function

$$x = f(l, k) \quad l \geq 0,\, k \geq 0$$

[1] Note that minimizing the cost of producing the chosen level of output is a necessary but not a sufficient condition for profit maximization.

which identifies the maximum output that can be produced with any non-negative input combination. The old guy told you the production function.

Visualize this production function in two-dimensional space.
For example, if $x = l^{.2}k^{.8}$



Stated mathematically, if the production manager knows the production function, the production manager's problem is to

$$\min_{wrt\ l \text{ and } k} e = wl + rk$$

subject to

$$f(k, l) = x$$

That is, the production manager wants to minimize expenditures e, subject to a number of constraints. The solution to this problem is the conditional input demand functions for labor and capital

$$l_c = l_c(x, w, r)$$

and

$$k_c = k_c(x, w, r)$$

These two demand identify the amount of labor and capital the production manager wants to hire to minimize the cost of producing x units of output given input prices w and r. They are called *conditional* because they are conditional on the output level.

What then is

$$e^* = wl_c(x, w, r) + rk_c(x, w, r) = c(x, w, r)$$

It is the firm's cost function

Below, we will assume explicit forms for the production function (starting with the simple Cobb-Douglas) and for each assumed production function derive the firm's cost function by solving the production manager's problem.

Note that one can also go the other direction; that is, with knowledge of the firm's cost function one has sufficient information to derive the firm's production function. A production function and a cost function are just two different ways of conveying the same information.

Deriving the Cobb-Douglas cost function from the Cobb-Douglas production function

$$x = f(k, l) = al^{\alpha}k^{1-\alpha}$$

The production manager's problem is to

$$\min_{wrt\ l \text{ and } k} e = wl + rk$$

subject to

$$x = f(k, l) = al^{\alpha}k^{1-\alpha}$$

Looking ahead, let's think about how we might derive the conditional input demand functions for labor and capital (solve this constrained optimization problem). Consider a special case where $x = al^{.5}k^{.5}$, Solve the constraint for k, the solution is:$k = \frac{1}{a^2 l}x^2$.[2] Substitute this into the objective function to obtain

$$wl + r(\frac{1}{a^2 l}x^2)$$

The problem is now one of minimizing this function wrt the single variable l. Take the partial derivative wrt l, $D_l(wl + r(\frac{1}{a^2 l}x^2)) = w - \frac{1}{a^2 l^2}rx^2$. Set this equal to zero and solve for l to get $w - \frac{1}{a^2 l^2}rx^2 = 0$, Solution is: $l_c(x, w, r) = \frac{1}{aw}\sqrt{rwx^2} = \frac{x}{aw}\sqrt{rw}$, which is the conditonal demand function for labor. By symmetry, the conditional demand function for capital will be $k_c(x, w, r) = \frac{x}{ar}\sqrt{rw}$.

Taking it one step further, the cost function is

$$\begin{aligned} c(x, w, r) &= wl_c(x, w, r) + rk_x(x, w, r) \\ &= w\frac{x}{aw}\sqrt{rw} + r\frac{x}{ar}\sqrt{rw} \\ &= \frac{2}{a}x\sqrt{rw} \end{aligned}$$

[2] This function is a rearrangement of the production function. It identifies the amount of k that must be hired given x and l.

So, in conclusion,$c(x, w, r) = \frac{2}{a}x\sqrt{rw}$, and $x = f(k, l) = al^{.5}k^{.5}$, are two different ways to describe the same technology for producing widgets. In this example both represent the Cobb-Douglass technology where $\alpha = \beta = .5$.

For fun do you think you can start with this cost function,$c(x, w, r) = \frac{2}{a}x\sqrt{rw}$, and derive $x = f(k, l) = al^{.5}k^{.5}$.

Doing so will require the use of something called *Shepard's Lemma.*

What is a lemma? A small animal that commits suicide by running into the ocean.

Consider the general cost function

$$c = c(x, w_1, w_2, ..., w_N)$$

Shepard's lemma says

$$\frac{\partial c(x, w_1, w_2, ..., w_N)}{\partial w_i} = l_{ci} = l_{ci}((x, w_1, w_2, ..., w_N)$$

where $l_{ci}((x, w_1, w_2, ..., w_N)$ is the conditional demand function for input i. So, for example

$$\frac{\partial c(x, w, r)}{\partial r} = k_c = k_c(x, w, r)$$

Shepard's Lemma is very useful.

For example, consider the cost function we derived above

$$c(x, w, r) = \frac{2}{a} x \sqrt{rw}$$

In this case, we already derived the conditional input demand functions above, but, for fun, assume we don't know them. Applying Shepard's Lemma$D_w(\frac{2}{a} x \sqrt{rw}) = \frac{1}{a} r \frac{x}{\sqrt{rw}}$

$$\frac{\partial c(x, w, r)}{\partial w} = l_c(x, w, r) = \frac{1}{a} r \frac{x}{\sqrt{rw}}$$

and

$$\frac{\partial c(x, w, r)}{\partial r} = k_c(x, w, r) = \frac{1}{a} w \frac{x}{\sqrt{rw}}$$