

1 Economic Examples of Partial Derivatives

partialeg.tex April 3, 2012

Let's start with production functions. A production function is one of the many ways to describe the state of technology for producing some good/product. Assume the firm produces a single output, x , using two inputs, labor, l , and capital k . The production function

$$x = f(k, l) \quad l \geq 0, k \geq 0$$

identifies the maximum output, x , that can be produced with any nonnegative combination of labor and capital.

Visualize this production function in two-dimensional space. Physics requires that $f(0, 0) = 0$.

- Note that a production function is only one of many ways to describe the state of technical knowledge for producing a good
- The production function can be used to identify the input requirement set

$$\{(k, l) : f(k, l) \geq x, l \geq 0, k \geq 0\}$$

- the production possibilities set X , where X is the set of all technically feasible output-input combinations, (x, k, l) . Note that X is a subset of R_3^+ , where R_n^+ denotes n -dimensional nonnegative space.

$$\{(x, k, l) : f(k, l) \geq x, l \geq 0, k \geq 0\}$$

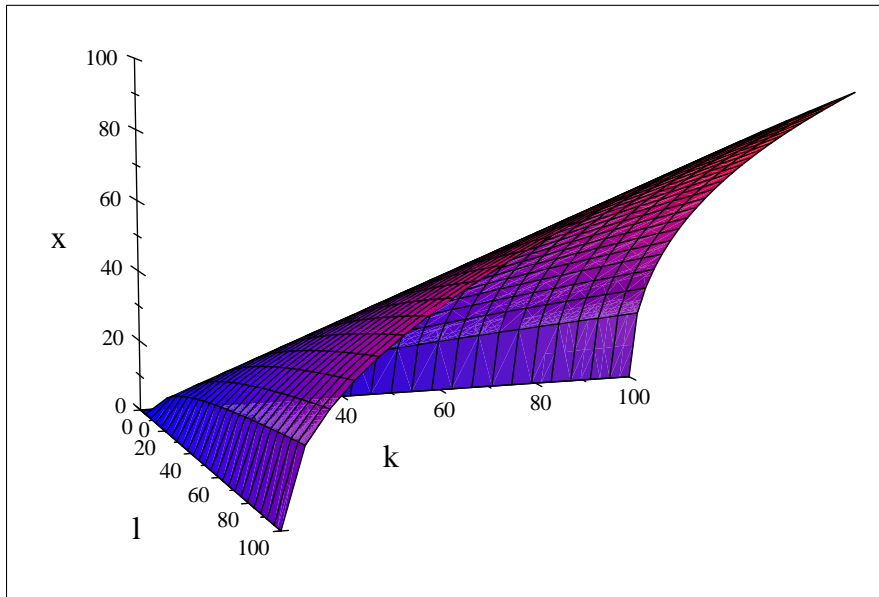
- the cost function, $c(x, w, r)$, is another way to represent the state of technical knowledge for producing x . I will show that later.

Consider the specific production function

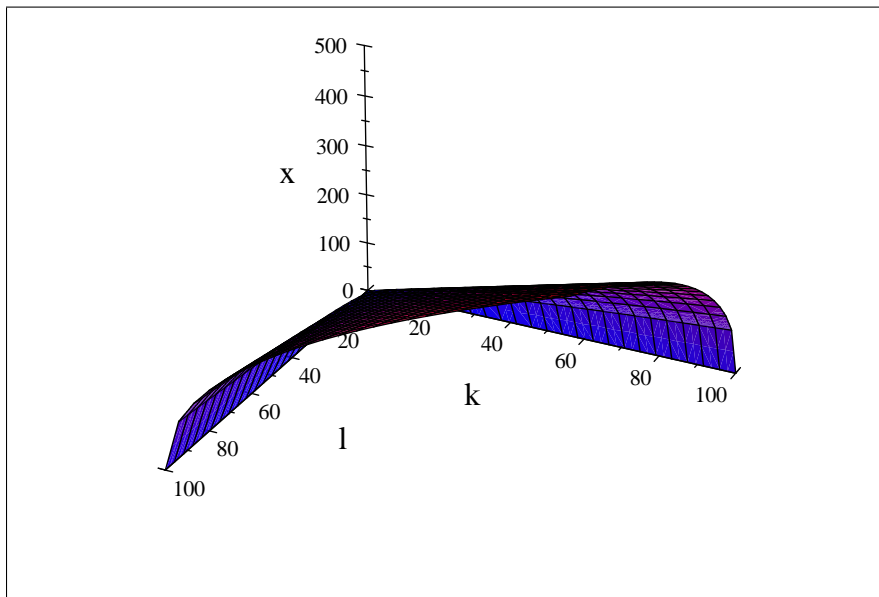
$$x = f(k, l) = al^\alpha k^{1-\alpha}$$

where $a > 0$ is a constant and $0 < \alpha < 1$ is another constant. This particular production function is called the Cobb-Douglas production function with constant returns to scale.¹ More generally, without CRTS it is $f(k, l) = al^\alpha k^\beta$, $a, \alpha, \beta > 0$

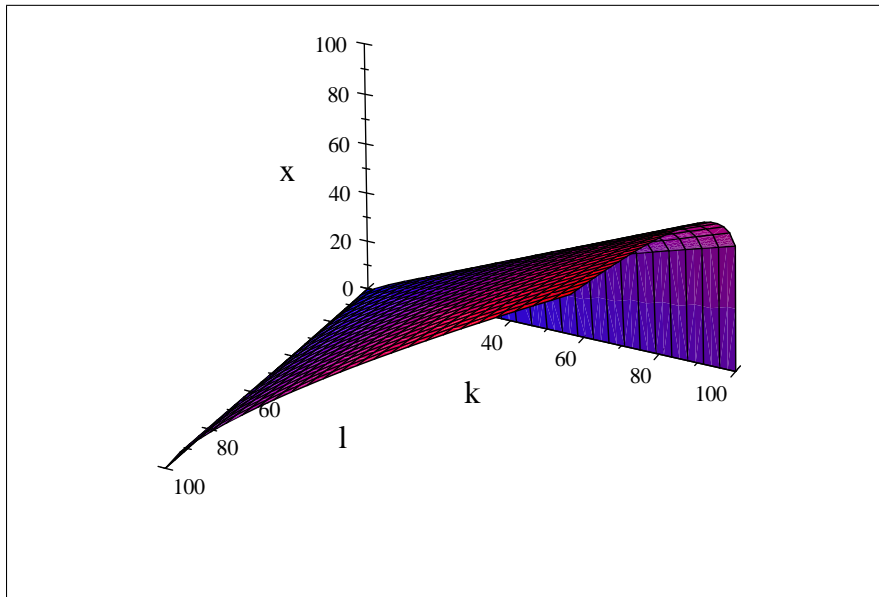
¹In the past, but not so much any more, the CD form was chosen to represent an economy's aggregate production function. See http://en.wikipedia.org/wiki/Cobb%E2%80%93Douglas_production_function



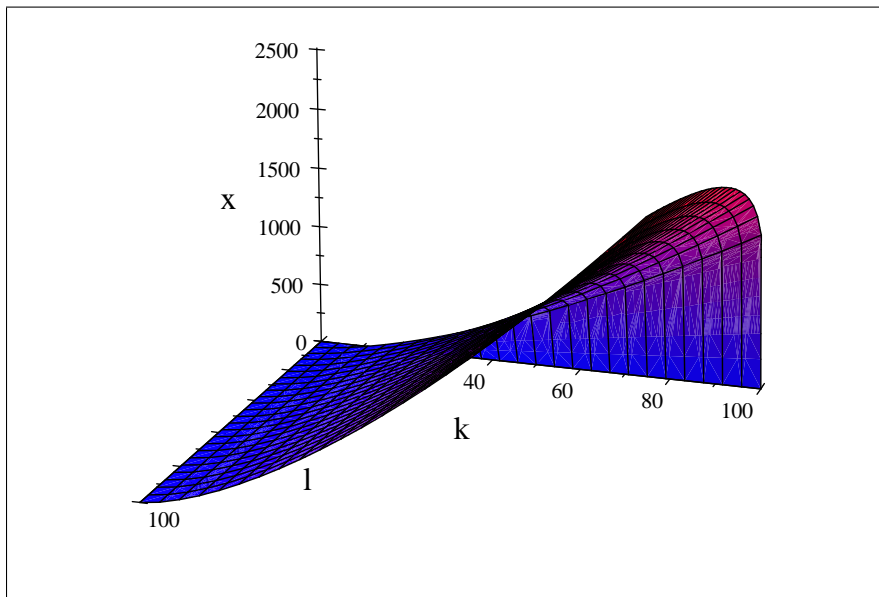
Cobb-Douglas with $a=1$, $\alpha=\beta=.5$



Cobb-Douglas with $a=5$, $\alpha=\beta=.5$



Cobb-Douglas with $a=1$, $\alpha=.8$, $\beta=.2$



Cobb-Douglas with $a=1$, $\alpha=1.5$, $\beta=.2$

Paul Douglass was an economist at the University of Chicago. He went on to be a famous Senator from Illinois. Charles Cobb was a mathematician with an office down the hall from Douglass, or so goes the story. Douglass wanted to

specify a production function that had the kind of isoquants one draws on the board in basic courses in microeconomics, but he did not know how to do it. So, he walked down the hall past the math offices until he found an open door, and Cobb showed him a functional form that fit his bill.

The partial derivatives of a production function have fancy economic names:

Definition: The marginal product of labor is

$$mp_l(k, l) = \frac{\partial f(k, l)}{\partial l}$$

For the Cobb-Douglas with CRTS, The marginal product of labor is

$$mp_l(k, l) = \frac{\partial f(k, l)}{\partial l} = a\alpha l^{\alpha-1} k^{1-\alpha}$$

For this Cobb-Douglas, is the marginal product of labor always positive?

Definition: The marginal product of capital is

$$mp_k(k, l) = \frac{\partial f(k, l)}{\partial k}$$

For this Cobb-Douglas, The marginal product of capital is

$$mp_k(k, l) = \frac{\partial f(k, l)}{\partial k} = a(1 - \alpha)l^\alpha k^{-\alpha}$$

For this Cobb-Douglas, it is also always positive.

For this Cobb-Douglas, what happens to the marginal product of labor as labor increases?

$$\frac{\partial(mp_l(k, l))}{\partial l} = \frac{\partial^2 f(k, l)}{\partial l^2} = (\alpha - 1)a\alpha l^{\alpha-2} k^{1-\alpha} < 0 \text{ Why is it always negative?}$$

That is, the marginal product of labor, while always positive, always decreases as l increases. The marginal product of capital also always decreases as k increases.

$$\frac{\partial(mp_k(k, l))}{\partial k} = \frac{\partial^2 f(k, l)}{\partial k^2} = -\alpha a(1 - \alpha)l^\alpha k^{-\alpha-1} < 0$$

For this Cobb-Douglas, the *law of diminishing marginal productivity* applies. Said generally, the law applies to $f(k, l)$ with respect to both inputs if (this is a sufficient but not necessary condition)

$$f_{ll}(k, l) < 0 \text{ and } f_{kk}(k, l) < 0$$

- Note that the *law* does not require the marginal products to always be positive. The Cobb-Douglas form is sufficient but not necessary for the *law* to hold.
- Not that the law is not really a law. It more something that we often, but not always, observe.

For this Cobb-Douglas production function what happens to the marginal product of labor (capital) as the quantity of capital (labor) increases. Does the marginal product of labor increase when k increases?

$$\frac{\partial(mpl(k, l))}{\partial k} = \frac{\partial^2 f(k, l)}{\partial l \partial k} = (1 - \alpha) a \alpha l^{\alpha-1} k^{-\alpha} > 0$$

How do we know it is always positive? As k increases, labor becomes more productive.

Do you think the Cobb-Douglas production function is an accurate description of the state of technical knowledge for producing any real-world products? No.

- We study it because it is simple, not because it is realistic.

For more on the Cobb-Douglas production function see your micro text or <http://en.wikipedia.org/wiki/Cobb-Douglas>

Cobb C W and Douglas P H (1928) "A Theory of Production", American Economic Review, 18 (Supplement), 139-165.

For an interesting interactive applet on the Cobb-Douglas production function, go to

<http://www.fgn.unisg.ch/eurmacro/tutor/cobb-douglas.html>

Think about some other functional forms for 2-input production functions.

Typically firms use more than two inputs. In which case, one might represent the production function as

$$x = f(l_1, l_2, \dots, l_N)$$

where l_i is the quantity of input i .

Can you specify an N input Cobb-Douglas Production Function.

The production functions that economists estimate are typically much more complicated than this simple Cobb-Douglas form. Before computers, we were restricted to simple functional forms such as the Cobb-Douglas, but this is no longer the case.

There are lots more economic examples of partial derivatives in the review questions.

One of the problems with marginal product functions is that the marginal product is sensitive to the units in which x , l , and k are measured. For example, changing the output from pounds of stuff to tons of stuff will drastically change the numerical values of the marginal products. It would be nice to be able to denote the productivity of the a marginal unit of input in a way that is not sensitive to the units that the inputs and output are measured.

Can we come up with such a measure? Yes. They are called elasticities

2 Looking ahead, using partial derivatives to find the a maximum.

Imagine a mountain in the room $y = f(x_1, x_2)$. we want to find the values of x_1 and x_2 that maximize y .

Consider two individuals: Lewis and Clark. Both are smart and logical. But, they both have perception disabilities. Lewis can only see and perceive in the directions North and South.

Clark has the same problem but he is an east-west kind of guy. It is, as if east west does not exist for Lewis and north south does not exist for Clark.

Both guys can wander around in both dimensions, but both lack shortterm memory; that is, they can't remember what it was like the last place they were at.

Each of you pick a partner. You are Lewis and Clark.

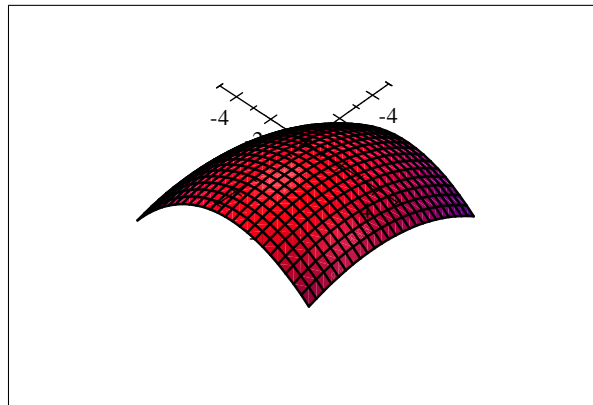
Your job is to write down a rule that you can use to find the values of x_1 and x_2 that max y

The rule: keep moving and looking until each of you is at a point that is a max in your direction, and you are both standing on the same point. That point will likely be a max, but not for sure. (maybe not the global max).

Could you use partial derivatives to find this point? At the max $\frac{\partial f(x_1^0, x_2^0)}{\partial x_1} = 0$, and $\frac{\partial f(x_1^0, x_2^0)}{\partial x_2} = 0$.

Use partial derivatives to find the values of x_1 and x_2 that max

$$f(x_1, x_2) = -2x_1^2 - x_2^2 + 4x_1 + 4x_2 - 3$$



$$\frac{\partial f(x_1, x_2)}{\partial x_1} = -4x_1 + 4$$

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = -2x_2 + 4$$

find the values of x_1 and x_2 where both of these partials are zero.

That is solve

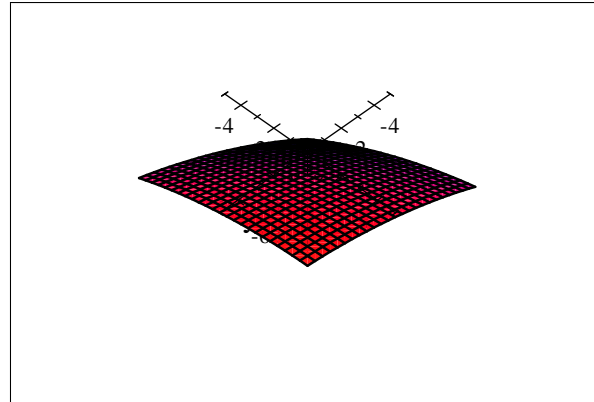
$$-4x_1 + 4 = 0$$

$$-2x_2 + 4 = 0$$

In this simple case each can be solved separately. $x_1^0 = 1$ and $x_2^0 = 2$

Use partial derivative to find a likely candidate for those values of x_1 and x_2 that maximize

$$f(x_1, x_2) = -2x_1^2 - 2x_1x_2 - 2x_2^2 + 36x_1 + 42x_2 - 158$$



$$\frac{\partial f(x_1, x_2)}{\partial x_1} = -4x_1 - 2x_2 + 36$$

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = -2x_1 - 4x_2 + 42$$

$$\begin{aligned} -4x_1 - 2x_2 + 36 &= 0 \\ -2x_1 - 4x_2 + 42 &= 0 \end{aligned}, \text{ Solution is: } \{x_1 = 5, x_2 = 8\}$$