1 Partial Derivatives

April 2, 2012

While derivatives are of interest to economists, as our previous examples show, they are often not applicable because most of the mathematical functions used to describe economic concepts are multivariate functions.

E.g., we assumed above that cost was soley a function of output, so specified the cost function as $c = c(x)$. But, more generally, minimum cost is not just a function of how many units of output the firm chooses to produce but also a function of input prices.

Consider the cost function for a firm that uses two inputs, l and k (labor and capital) which it purchases in a competitive market at the prices w and r, where r is the rental price of capital. The firm's cost function indentifies the minimum cost of producing x units out output at prices w and r. That is

$$c = c(x, w, r)$$

In the simple form $c(x)$, we were implicitly holding w and r constant at some specific amounts. Cost more generally is a function of the output level and the input prices. Note that for a firm that buys inputs in a competitive market w and r are exogenous.

Can we take the derivative of $c = c(x, w, r)$ with respect to x, w or r? Strictly speaking NO.

However, one can take the *partial derivative* of the cost function with respect to any of the three variables.

When one partially differentiates a function of several variables with respect to one of those variables, one treats the other variables as constants and uses the rules for differentiating functions of one variable.

We denote the partial derivative of $c(x, w, r)$ with respect to x as

$$\frac{\partial c(x, w, r)}{\partial x}$$

Note the different notation for a partial derivative: ∂ not d.

Further note that the notation $c'(x, w, r)$ is ambiguous so is replaced with $c_x(x, w, r)$ for $\frac{\partial c(x,w,r)}{\partial x}$, $c_w(x, w, r)$ for $\frac{\partial c(x,w,r)}{\partial w}$ and $c_r(x, w, r)$ for $\frac{\partial c(x,w,r)}{\partial r}$

Putting all this a little more formally and generally. Consider the function

$$y = f(x_1, x_2, ...x_N) = f(\mathbf{x})$$

where a bolded x, $\mathbf{x}$ denotes a vector of x's.

Consider the simpliest case

$$y = f(x_1, x_2) = f(\mathbf{x})$$

What is the partial derivative of $f(x_1, x_2)$ with respect to x_1 evaluated at (x_1^0, x_2^0)?

definition

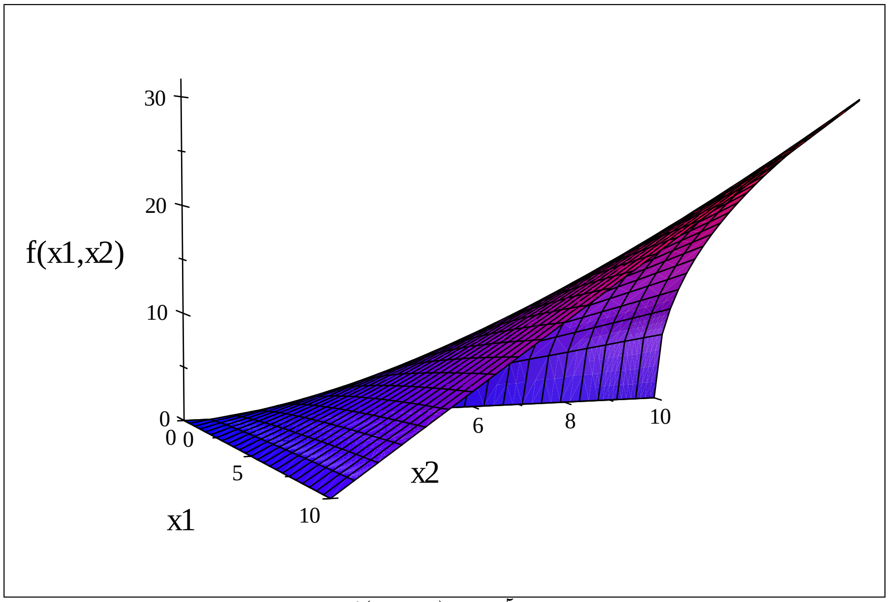
$$\frac{\partial f(x_1^0, x_2^0)}{\partial x_1} = f_{x_1}(x_1^0, x_2^0) = \lim_{t \to 0} \frac{f(x_1^0 + t, x_2^0) - f(x_1^0, x_2^0)}{t}$$

What is the partial derivative of $f(x_1, x_2)$ with respect to x_2 evaluated at (x_1^0, x_2^0)?
definition

$$\frac{\partial f(x_1^0, x_2^0)}{\partial x_2} = f_{x_2}(x_1^0, x_2^0) = \lim_{t \to 0} \frac{f(x_1^0, x_2^0 + t) - f(x_1^0, x_2^0)}{t}$$

Can the partial derivative be described as the slope of a tangent line to the function at a point?
Show it graphically in room

Consider the following function $f(x_1, x_2) = x_1^{.5} x_2$



$f(x_1, x_2) = x_1^{.5} x_2$

How would you graphically indentify the partial derivatives in terms of this graph.

Derriving a partial derivative from scratch, a simple example. Assume

$$y = f(x_1, x_2) = ax_1 x_2$$

In which case

$$\frac{\partial f(x_1^0, x_2^0)}{\partial x_1} = f_{x_1}(x_1^0, x_2^0) = \lim_{t \to 0} \frac{a(x_1^0 + t)x_2^0) - ax_1^0 x_2^0}{t}$$

$$= \lim_{t \to 0} \frac{ax_1^0 x_2^0 + atx_2^0 - ax_1^0 x_2^0}{t} = \lim_{t \to 0} \frac{atx_2^0}{t} = \lim_{t \to 0} ax_2^0 = ax_2^0$$

As with derivatives, we could use our basic definition of a partial derivative to find partial derivatives.

But, in general, if we are taking the derivative of a multivariate function with respect to one variable, we can treat the others as constants and use our derived rules for differentiating a function of a single variable.

E.g. find the derivatives of

$$z = f(x, y) = x + 2x^{1/2}y^{1/2} + y; \quad x \geq 0,\, y \geq 0$$

$$f_x = 1 + (1/2)(2)x^{-1/2}y^{1/2} + 0 = 1 + x^{-1/2}y^{1/2}$$
$$f_y = 1 + x^{1/2}y^{-1/2}$$