# 1 Optimization

optim-start.tex April 26, 2012

## 1.1 A start

I am going to start by solving some problems that we are familiar with. I will add theory after we get a sense of the issues involved in optimization. We currently have enough tools to solve a lot of maximization and minimization problems in economics.

Because resources are scarce, all optimization problems in economics are problems of constrained optimization: maximizing or minimizing some objective function subject to one or more constraints. E.g. maximizing profits subject to the state of technical knowledge and exogenous prices or the exogenous demand function for the firm's product.

Minimizing the cost of producing some exogenous level of output subject to the state of technical knowledge and input prices (the production manager's problem).

Finding the bundle of goods that maximizes an individual's utility subject to his or her budget constraint.

Le't express each of these problems in mathematical notation.

### 1.1.1 The profit maximization problem for the competitive firm

The competitive firm takes the state of technical knowledge as exogenous as well as input prices ($w$ and $r$) and the output price, $p$. The state of technology for producing the firm's product can be represented with either a production function or a cost function.

**In terms of the cost function, the firm's problem is to choose**

$$\max_{wrt\ x} \pi(x) = px - c(x, w, r)$$

The solution is of the form

$$x^s = x^s(p, w, r)$$

which is the firm's suppy function. The supply function indentifies the amount of output the firm want to produce (and sell) to maximize its profits, given $p$,

$w$, and $r$. The functional form of the supply function is completely determined by the functional form of the cost function (the state of technical knowledge).

Note that there are no explicit side constraints. Rather the constraints are embedded in the objective function.

**In terms of the production function, the firm's profit max problem is to choose**

$$\max_{wrt\ k\ l} \pi(k, l) = pf(k, l) - wl - rk)$$

The solution is of the form

$$l^d = l^d(p, w, r)$$

$$k^d = k^d(p, w, r)$$

These are the firm's input demand functions.[1] They identify the amounts of labor and capital the firm will want to hire to maximize its profits as a function of $p$, $w$, and $r$. The functional forms of these demand functions are completely determined by the functional form of the production function (the state of technical knowledge).

Note that there are no explicit side constraints. Rather the constraints are embedded in the objective function.

### 1.1.2 The consumer's problem

$$\max_{wrt\ x_1, x_2} u(x_1, x_2)$$

Subject to

$$y \geq p_1 x_1 + p_2 x_2$$

If one assumes that more is always preferred to less, the consumer will exhaust his budget and

$$y = p_1 x_1 + p_2 x_2$$

Solving the budget constraint for $x_2$ on obtains

$$x_2 = \frac{y}{p_2} - \frac{p_1}{p_2} x_1$$

Note that the equation for the budget constraint is always the same. Plugging it, in terms of $x_2$, into the utility function one obtains

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[1] Note how these are different from conditional input demand functions.

$$\max_{wrt\ x_1} u(x_1, \frac{y}{p_2} - \frac{p_1}{p_2}x_1)$$

making utility a function of only one choice variable, $x_1$

The solution to the consumer's problem is the demand functions

$$x_1^d = x_1^d(y, p_1, p_2)$$
$$x_2^d = x_2^d(y, p_1, p_2)$$

Let's find the consumer's demand functions for $x_1$and $x_2$ assuming $u = x_1 x_2$

$$\max_{wrt\ x_1, x_2} u(x_1, x_2) = x_1 x_2$$

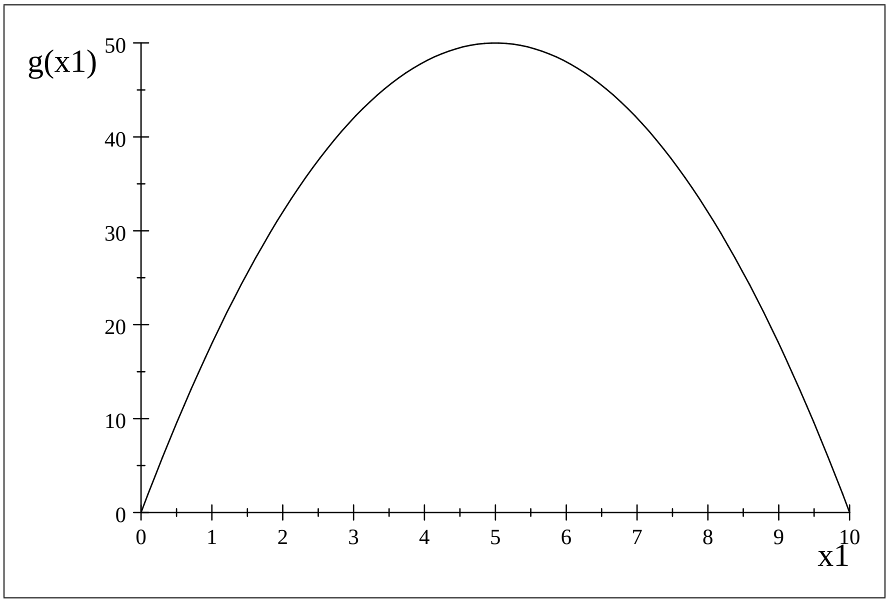
subject to

$$x_2 = \frac{y}{p_2} - \frac{p_1}{p_2}x_1$$

Substituting the constraint into the objective function, one obtains

$$\max_{wrt\ x_1} = g(x_1) = x_1 \left[ \frac{y}{p_2} - \frac{p_1}{p_2} x_1 \right] = \frac{y}{p_2} x_1 - \frac{p_1}{p_2} x_1^2$$

The following is an example of $g(x_1)$ when $y = 100,\ \ p_1 = 10$ and $p_2 = 5$,



$g(x_1) = \frac{y}{p_2} x_1 - \frac{p_1}{p_2} x_1^2$ when $y = 100$, $p_1 = 10$, $p_2 = 5$

Note that we have turned an constrained optimization in two variables into an unconstrained optimization problem in one variable.

Also note that the $x_1$ that maximizes utilty is at a point where $\frac{\partial g(x_1)}{\partial x_1} = 0$; that is, at a *critical point.*

The solution is the consumer's demand function for good 1,

$$x_1^d = x_1^d(y, p_1, p_2)$$

How do we find the value of vector $\boldsymbol{x}$, $\boldsymbol{x}^d$, that maximizes utility subject to the constraints?

We are looking for the top of the utility mountain. How can we find it?

At the top of a twice-differentiable mountain, $g'(x_1) = 0$. We call the value of $x_1$, where $g'(x_1) = 0$ a critical value of $x_1$and denote it $x_1^0$. That is, $g'(x_1^0) = 0$.

Find $x_1^0$

$$g'(x_1) = \frac{y}{p_2} - \frac{2p_1}{p_2}x_1$$

Setting this equal to zero and solving for $x_1$, one obtains

$$\begin{aligned} 0 &= \frac{y}{p_2} - \frac{2p_1}{p_2}x_1 \\ &\Rightarrow \frac{2p_1}{p_2}x_1 = \frac{y}{p_2} \\ &\Rightarrow x_1^0 = x_1(y, p_1, p_2) = \frac{yp_2}{p_2 p_1 2} = \frac{y}{2p_1} \end{aligned}$$

So, $x_1^0 = x_1(y, p_1, p_2) = \frac{y}{2p_1} = \frac{.5Y}{p_1}$ is possibly the demand function for good one. If it is, it says that the individual should spend 1/2 of his or her income on each good.

How would you make sure that $x_1^0$ is the quantity of $x_1$ that maximizes rather than minimizes utility.?

Maybe look at the second derivative of $g(x_1)$, $g''(x_1)$ at $x_1^0$.

$$g^{''}(x_1) = -\frac{2p_1}{p_2}$$

We want to evaluate this at $x_1^0$. Note that it is the same constant $x_1^0$ at every value of $x$, so $g^{''}(x_1^0) = -\frac{2p_1}{p_2} < 0 \; \forall p_i > 0$.

What does this tell us? that utility is maximized at

$$x_1^0 = x_1(y, p_1, p_2) = \frac{y}{2p_1}$$

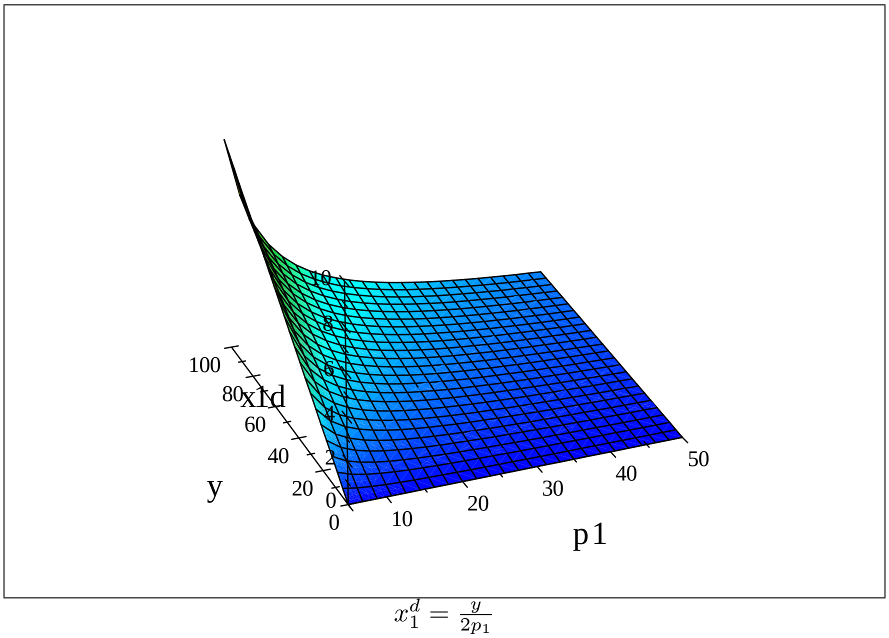
Why?

The individual's demand functions (Marshallian demand functions) are

$$x_1^d = x_1^d(y, p_1, p_2) = \frac{y}{2p_1}$$

$$x_2^d = x_2^d(y, p_1, p_2) = \frac{y}{2p_2}$$

The individual will spend 1/2 of his income on each good. This should not surprise us given the symmetrical way the two goods enter the demand function.The graph is the the demand function for $x_1$.



$x_1^d = \frac{y}{2p_1}$

Now find Wilber's demand functions for $x_1$ and $x_2$ assuming his utility function is $u = u(x_1, x_2) = ax_1 + bx_2$ where $a$ and $b$ are both positive.

Now try and solve for the demand function for $x_1$ assuming

$$u(x_1, x_2) = ax_1^{\alpha} x_2^{1-\alpha} \quad \text{where } 1 > \alpha > 0,\ a > 0$$

Begin by solving the budget constraint for $x_2$ in terms of $x_1$ which we know is

$$x_2 = \frac{y - p_1 x_1}{p_2}$$

substitute this into the utility function to get

$$u = u(x_1) = ax_1^{\alpha}(\frac{y - p_1 x_1}{p_2})^{1-\alpha}$$

The demand function for $x_1$ is that $x_1$ that maximizes $u(x_1)$. Start by looking for critical values of $x_1$ that is, values of $x_1$ where $\frac{\partial u(x_1)}{\partial x_1} = 0$

$$\begin{aligned}
\frac{\partial u(x_1)}{\partial x_1} &= \frac{\partial[ax_1^{\alpha}(\frac{y - p_1 x_1}{p_2})^{1-\alpha}]}{\partial x_1} \\
&= \alpha a x_1^{\alpha-1}(\frac{y - p_1 x_1}{p_2})^{1-\alpha} - (1-\alpha) a x_1^{\alpha}(\frac{y - p_1 x_1}{p_2})^{-\alpha}(\frac{p_1}{p_2}) \\
&= a x_1^{\alpha}(\frac{y - p_1 x_1}{p_2})^{-\alpha}[\alpha a x_1^{-1}(\frac{y - p_1 x_1}{p_2}) - (1-\alpha)(\frac{p_1}{p_2})]
\end{aligned}$$

So, this is the slope of the utility function in terms of $x_1$ with the budget constraint incorporated. All we have to do to find a the critial point or points is to set it equal to zero and solve for the$x_1^0$.

Set the partial to zero: $ax_1^{\alpha}(\frac{y-p_1x_1}{p_2})^{-\alpha}[\alpha a x_1^{-1}(\frac{y-p_1x_1}{p_2})-(1-\alpha)(\frac{p_1}{p_2})]=0$. Simplify by multiplying both side by $ax_1^{\alpha}(\frac{y-p_1x_1}{p_2})^{-\alpha}$ to get $[\alpha a x_1^{-1}(\frac{y-p_1x_1}{p_2})-(1-\alpha)(\frac{p_1}{p_2})]=0$. This can be further simplifed by multiplying through by $p_2$ to get $\alpha a x_1^{-1}(y-p_1x_1)-(1-\alpha)p_1=0$

So, to get my math program can deal with this, I will make $x_1$ just $x$, and $p_1$ just $p$.

$$\alpha a x^{-1}(y-px)-(1-\alpha)p=0, \text{Solution is: } \begin{cases} \left\{-ay\frac{\alpha}{-p+p\alpha-ap\alpha}\right\} & \text{if} \quad a\neq 0\wedge\alpha\neq 0\wedge p-p\alpha+ap\alpha\neq 0 \\ \mathbb{C}\setminus\{0\} & \text{if} \quad y=0\wedge a\neq 0\wedge\alpha\neq 0\wedge p-p\alpha+ap\alpha= \\ \emptyset & \text{if} \quad a\neq 0\wedge y\neq 0\wedge\alpha\neq 0\wedge p-p\alpha+ap\alpha= \\ \mathbb{C} & \text{if} \quad a=0\wedge p=0\wedge\alpha=1 \\ \mathbb{C} & \text{if} \quad p=0\wedge\alpha\neq 1\wedge a\alpha=0 \\ \mathbb{C} & \text{if} \quad a=0\wedge p\neq 0\wedge\alpha=1 \\ \emptyset & \text{if} \quad p\neq 0\wedge\alpha\neq 1\wedge a\alpha=0 \end{cases}$$

So it looks like the critical value of $x_1$ is

$$\begin{aligned} x_1^0 &= -ay\frac{\alpha}{-p+p\alpha-ap\alpha} \\ &= \frac{\alpha a y}{(p-p\alpha+ap\alpha)} \\ &= \frac{\alpha a y}{(p-p\alpha+ap\alpha)} \\ &= \frac{\alpha a y}{p_1(1-\alpha+a\alpha)} \end{aligned}$$

This looks like it might be the demand function for good 1 given the utility function $ax_1^{\alpha}x_2^{1-\alpha}$

### 1.1.3 Remember the production manager's problem

Stated mathematically, the production manager's problem is to

$$\min_{wrt\ l \text{ and } k} e = wl + rk$$

subject to

$$x = f(k,l)$$

That is the production manager wants to minimize expenditures $e$, subject to a number of constraints. The solution to this problem is the conditional input demand functions for labor and capital

$$l_c^d = l_c^d(x, w, r)$$

and

$$k_c^d = k_c^d(x, w, r)$$

These two demand identify the amount of labor and capital the production manager want to hire to minimize the cost of producing $x$ units of output given input prices $w$ and $r$. They are called *conditional* because they are conditional on the output level.

What then is

$$e^* = wl_c^d(x, w, r) + rk_c^d(x, w, r)?$$

It is the firm's cost function, $c(x, w, r)$

For example, if one assumes the simpleCobb-Douglas production function

$$x = f(k, l) = al^\alpha k^{1-\alpha}$$

The production manager's problem is to

$$\min_{wrt\ l \text{ and } k} \quad e = wl + rk$$

subject to

$$x = f(k, l) = al^\alpha k^{1-\alpha}$$

How would you solve the production manager's problem assuming

$$x = f(k, l) = al^\alpha k^{1-\alpha}$$

Don't do it at this point, only write out the steps.

## 1.2 How to find the max of a function of a single variable: two questions

See chapter 8, Single-Variable Optimization, in Sydsaeter and Hammond.

These are the first two questions on the optimization set of review questions. Have them do it in groups in class.

1. Describe in general terms how one would search for the global max or min of a **twice-differentiable** function of one variable, $f(x)$, $a \leq x \leq b$. What are the potential pitfalls of assuming the max (min) is at a level of $x$, $x^o$, where $f_x(x^o) = 0$. Explain

   **Answer:** One needs to look everywhere for the global max. A necessary condition for global **interior** max is that $f(x^o) = 0$, so find all the values of $x$ such that $f_x(x^o) = 0$ - these are the *critical* values of $x$.

   Check second-order conditions and toss out all the candidates that are not local max; that is, toss all those critical values of $x$ that do not have $f_{xx}(x^o) \leq 0$. The remaining $x_o$ are all local interior max.

   The largest one (there might only be one) is the global interior maximum. Now check to make sure that the global maximimum is not one of the end points. That is, check the end points ($a$ and $b$) to see if the value of $f(x)$ at one of the end points is  is greater than the value of the function at the global interior max.  If one is, it is the global max; if not, the global interior max is the global max. (Note that if there is no limits on the range of $x$ there are no corners, and the global interior max is the global max because all values of $x$ are interior values.)

   As an aside, and assuming $a \leq x \leq b$, the "the global interior max might not be the highest point in the interior of $x$. In which case, none of the point(s) that are higher are the global interior maximum. (Can you show me a graphical example of this case?)

   What are the potential pitfalls of assuming the max (min) is at a level of $x$, $x^o$, where $f_x(x^o) = 0$. It might not be a local interior max. It could, for example, be a minimum. And, even if it is a local interior max, it might not be the global interior max. And, even if it is a global interior max, it might not be a global max.

   Further note that the interior max might be at a point where the function is not twice differentiable. In the question we assumed twice differentiability.

2. Continue to assume that $f(x)$ is twice differentiable. Assume that you have found an $x$, $x^o$, where $f_x(x^o) = 0$.Identify an additional condition that is sufficient, but not necessary, for $x^0$ to be a local interior max. Explain. Why isn't your condition necessary?

   **answer:** $f_{xx}(x) < 0 \ \forall x$ is one possibility. It is not necessary because could be a lots of $x^o$ that maximize a function but where $f_{xx}(x) \not< 0 \ \forall x$. Consider a case where $f_{xx}(x^0) < 0$ but $f_{xx}(x) \not< 0 \ \forall x$. Or consider a case, where at the local maximum, $f_x(x^o) = 0$ and $f_{xx}(x^0) = 0$. What is going on with this second case? Consider $f(x) = 5$.

Keep in mind that using software to graph $f(x)$ is typically a good way to start looking for the maximum and minimum

## 1.3 How to find the max of a function of two variables

See chapter 13, Multivariate Optimization, in Sydsaeter and Hammond.

Assume $y = f(x_1, x_2)$, $a \leq x_1 \leq b$, $c \leq x_2 \leq d$, is **twice differentiable**. Find the values of $x_1$ and $x_2$, $x_1^*$ and $x_2^*$, that maximize $y$. Conceptually, the steps are the same one follows when looking for the max of a function of only one variable. The only difference is that here one must check that the function is maximized in all 360° of directions of the $x_1 - x_2$ plane. Imagine you are standing at a point on $f(x_1, x_2)$, you have to look to make sure you are at a maximum in every direction, an infinite number of directions. In the case of $f(x_1)$, you only had to look in two directions.

First search for local interior maximium. A **necessary** condition for a local interior maximum at $x_1^0, x_2^0$ is that $f_{x_1}(x_1^0, x_2^0) = 0$ and $f_{x_2}(x_1^0, x_2^0) = 0$. All the points where this is true (there might be many) are critical points and potential candidates to be local interior maximums. Find all of these stationary points.

For the moment, assume there is only one stationary point, $x_1^0, x_2^0$. One needs to check second-order conditions to see if this point is a maximum, rather than, for example, a minimum. A **necessary** condition for the function to have a local interior max at $x_1^0, x_2^0$ is $f_{x_1x_1}(x_1^0, x_2^0) \leq 0$ and $f_{x_2x_2}(x_1^0, x_2^0) \leq 0$. That to be a interior maximum it must be a maximum in the $x_1$ directions and a maximum in the $x_2$ directions.

If you find a point, $x_1^0, x_2^0$ where

1. $f_{x_1}(x_1^0, x_2^0) = 0$
2. $f_{x_2}(x_1^0, x_2^0) = 0$
3. $f_{x_1x_1}(x_1^0, x_2^0) < 0$ and
4. $f_{x_2x_2}(x_1^0, x_2^0) < 0$

You have probably found a local interior maximum, but maybe not. You have found a point that is a local interior maximum in the $x_1$ direction and a point that is a local interior maximum in the $x_2$ direction. If so, it is probably a maximum in all of the off-diagonal directions as well, but maybe not: there are some complex functions where it is not.

Sufficient, but not necessary, conditions for function to have a local interior max at $x_1^0, x_2^0$ is

1. $f_{x_1}(x_1^0, x_2^0) = 0$
2. $f_{x_2}(x_1^0, x_2^0) = 0$

3. $f_{x_1x_1}(x_1^0, x_2^0) < 0$
4. $f_{x_2x_2}(x_1^0, x_2^0) < 0$ and
5. $f_{x_1x_1}(x_1^0, x_2^0) f_{x_2x_2}(x_1^0, x_2^0) - (f_{x_1x_2}(x_1^0, x_2^0))^2 > 0$

This last condition guarentees that the function is "enough" of a maximum in the $x_1$ and $x_2$ directions to make up for any goofyness in the the off-diagonal directions. For more details see page 467 in Sydsaeter and Hammond (note that their sufficient conditions are weaker than my sufficient conditions).

If there is only one local interior max, it is the global interior max, but is it the global max? The set feasible set of $x_1, x_2$ values is a rectangle. The global max could be at one the 4 corners ($(a, c)$, $(a, d)$, $(b, c)$, and $(b, d)$. Could it be on the edge of the rectange but not at a corner; that is, at a point where where one of the variables is at a corner but the other takes an interior value?

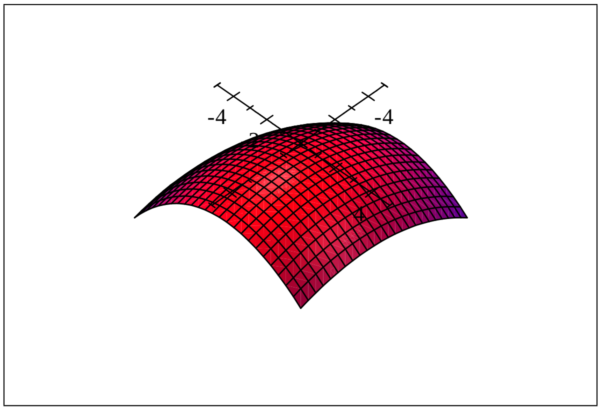
Remember that there might be more than one local interior maximum. If so, you need to find them all, and calculate the value of the function at each to determine which is the global interior maximum.

### 1.3.1 Some simple examples of two-variable functions

Use partial derivatives the find the values of $x_1$ and $x_2$ that max

$$f(x_1, x_2) = -2x_1^2 - x_2^2 + 4x_1 + 4x_2 - 3$$

A graph of the function is often very helpful, but only when the function to be maximized is a function two or fewer variables - with more variables than this, graphs fail us.



$-2x_1^2 - x_2^2 + 4x_1 + 4x_2 - 3$

$\frac{\partial f(x_1,x_2)}{\partial x_1} = -4x_1 + 4$
$\frac{\partial f(x_1,x_2)}{\partial x_2} = -2x_2 + 4$

find the values of $x_1$ and $x_2$ where both of these partials is zero.

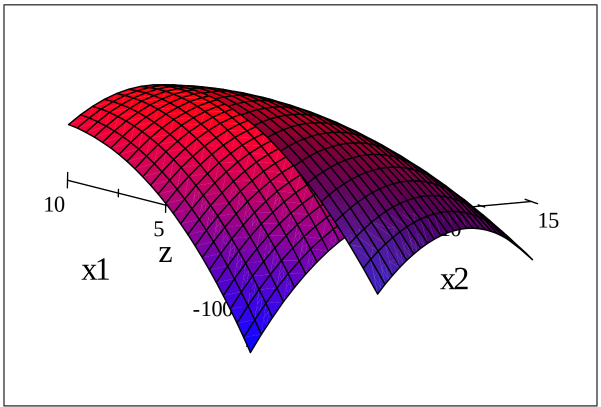
That is solve
$-4x_1 + 4 = 0$
$-2x_2 + 4 = 0$
In this simple case each can be solved separately. $x_1^0 = 1$ and $x_2^0 = 2$. So, $(1, 2)$ is a potential candidate for the interior max.

Check the second-order partial derivative. $D_{x_1x_1}(-2x_1^2 - x_2^2 + 4x_1 + 4x_2 - 3) = -4$ and $D_{x_2x_2}(-2x_1^2 - x_2^2 + 4x_1 + 4x_2 - 3) = -2$. Both of these are strictly negative for all values of $x_1$ and $x_2$, so we definitely have a maximum in the directions of the axis.

From the picture we know we have found the global max, but let's keep going. $D_{x_1x_2}(-2x_1^2 - x_2^2 + 4x_1 + 4x_2 - 3) = 0$. So $((-4)(-2) - 0^2) = 8 > 0$, so we have fulfilled the sufficient conditions for global max.

Use partial derivative to find a likely candidate for those values of $x_1$ and $x_2$ that maximize

$$f(x_1, x_2) = -2x_1^2 - 2x_1x_2 - 2x_2^2 + 36x_1 + 42x_2 - 158$$



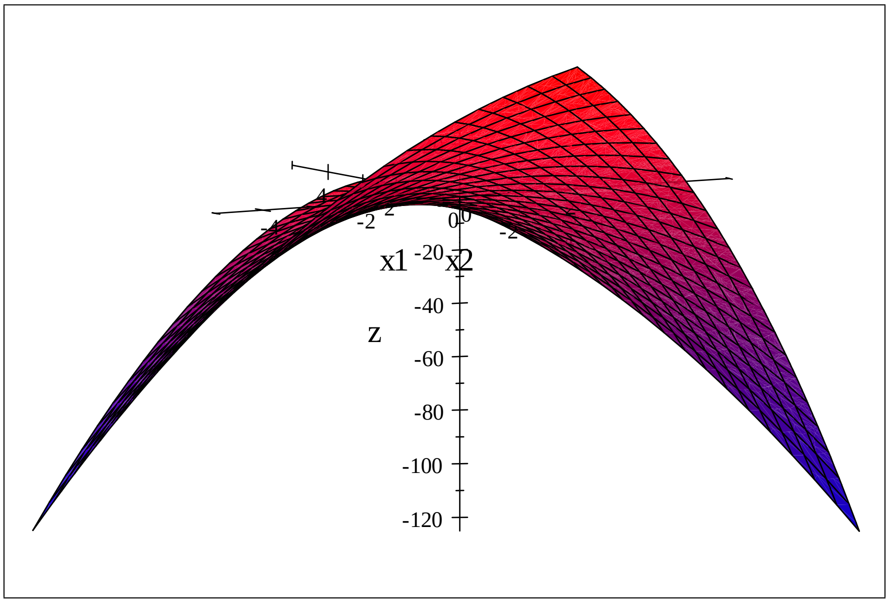
$-2x_1^2 - 2x_1x_2 - 2x_2^2 + 36x_1 + 42x_2 - 158$

$\frac{\partial f(x_1,x_2)}{\partial x_1} = -4x_1 - 2x_2 + 36$
$\frac{\partial f(x_1,x_2)}{\partial x_2} = -2x_1 - 4x_2 + 42$

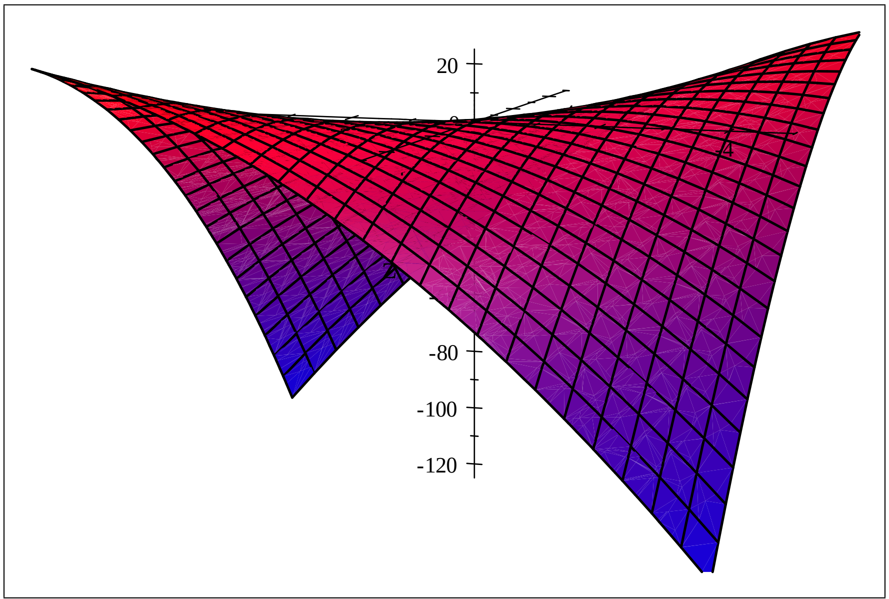
$\begin{array}{r} -4x_1 - 2x_2 + 36 = 0 \\ -2x_1 - 4x_2 + 42 = 0 \end{array}$ , Solution is: $\{x_1 = 5, x_2 = 8\}$. So, $(5, 8)$ is a stationary point.

Check the second-order conditions. $D_{x_1 x_1}(-2x_1^2 - 2x_1 x_2 - 2x_2^2 + 36x_1 + 42x_2 - 158) = -4$ , $D_{x_2 x_2}(-2x_1^2 - 2x_1 x_2 - 2x_2^2 + 36x_1 + 42x_2 - 158) = -4$ and $D_{x_1 x_2}(-2x_1^2 - 2x_1 x_2 - 2x_2^2 + 36x_1 + 42x_2 - 158) = -2$. The function is globally maximized at $(5, 8)$

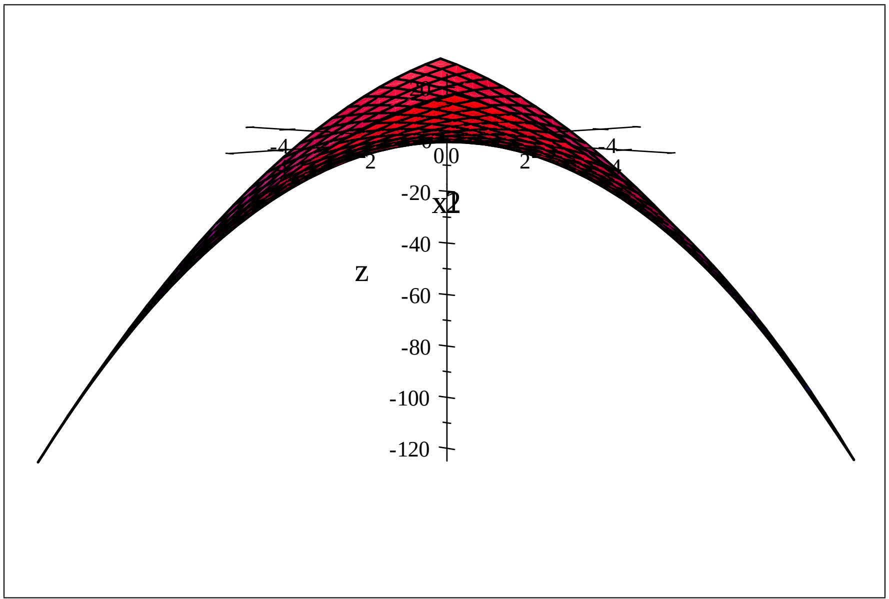
Now consider the function $f(x_1, x_2) = 3x_1 x_2 - x_1^2 - x_2^2$. Let's take a look at this function



$3x_1 x_2 - x_1^2 - x_2^2$



$3x_1x_2 - x_1^2 - x_2^2$



$3x_1x_2 - x_1^2 - x_2^2$

Looks to me like a function that has no local interior maximum. There is a stationary point; that is at point $x_1^0, x_2^0$ where $f_{x_1}(x_1^0, x_2^0) = 0$ and $f_{x_2}(x_1^0, x_2^0) =$

0 but this point is a maximum in one direction and a minimum in another direction.

Let's see what happens if we starting checking for a local interior maximum. First find the partial derivatives, $D_{x_1}(3x_1x_2 - x_1^2 - x_2^2) = 3x_2 - 2x_1$, and $D_{x_2}(3x_1x_2 - x_1^2 - x_2^2) = 3x_1 - 2x_2$. Set both of these equal to zero: $3x_2 - 2x_1 = 0$, $3x_1 - 2x_2 = 0$ and solve using substituion. Solving $3x_2 - 2x_1 = 0$, solution is: $x_2 = \frac{2}{3}x_1$. Substitute this into the other equation to get $3x_1 - 2(\frac{2}{3}x_1) = 0$, Solution is: $x_1^0 = 0$. Plugging this into either equation one finds out that $x_1^0 = 0$, so the only stationary point is at $(0, 0)$.

**need to check the following**
So far, all we know is that this is the only candidate for a local interior maxium. Now let's check the second-order conditions: $D_{x_1x_1}(3x_1x_2 - x_1^2 - x_2^2) = -2$ and $D_{x_2x_2}(3x_1x_2 - x_1^2 - x_2^2) = -2$ .

At this point, if we had not seen the graph of the function, we might mistakingly stopped and concluded the function is maximized at $(0, 0)$, but we know from the picture that is not maximized at this point. So, let's check the last part of the sufficient condition for a max, $f_{x_1x_1}(x_1^0, x_2^0)f_{x_2x_2}(x_1^0, x_2^0) - (f_{x_1x_1}(x_1^0, x_2^0))^2 > 0$. The cross-partial is $D_{x_1x_2}(3x_1x_2 - x_1^2 - x_2^2) = 3$, so $(-2)(-2) - 3^2 < 0$, which is not greater that zero, so the sufficient condition for at maximum at $(0, 0)$ is violated.