

# 1 What is a function?

February 5, 2012

## 1.1 What does it mean to say

$$y = f(x)$$

where  $x$  and  $y$  are variables

A function associates with each value of the variable  $x$  a unique value of the variable  $y$ .

Let's stick with the variables  $x$  and  $y$ , but keep in mind there is nothing special about those names. I could write  $t = h(m)$

$y = f(x)$  is a mapping from  $x$  to  $y$ : each value of  $x$  is mapped (transformed) by the function into a unique value of  $y$ , using some rule. If we write the function  $f(x)$   $f$  is the "name" of the rule.

The value that  $y$  takes for a given  $x$  is called the image of  $x$  (think of a goofy mirror where the reflection/image of  $x$  is not  $x$  but rather  $f(x)$ ).<sup>1</sup>

Another way to symbolically denote

$$y = f(x)$$

is

$$f : x \rightarrow y$$

which visually tells us that  $x$  values are mapped into  $y$  values, where the actual mapping is described by the rule  $f$ .

Contrast the  $f$  rule with some other rule

$$g : x \rightarrow y$$

We give different mapping rules different names. In the above, I have used the names  $f$  and  $g$  to denote different functions of  $y$  as a function of  $x$ . For example,  $f$  inf  $f(x)$  might denote the rule  $3x$  and  $g$  in  $g(x)$  denote the rule  $\ln x$ .

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<sup>1</sup>Note that all mirrors are goofy in that when you look in a mirror your left side appears on the right side of the image from the perspective of the image, and your right side appears on the left side of the image from the perspective of the image.

## 1.2 Some graphical examples of mappings and functions

- start with a linear function - first with two axis:  $x$  on the horizontal axis and  $y$  on the vertical axis
- then one line to another

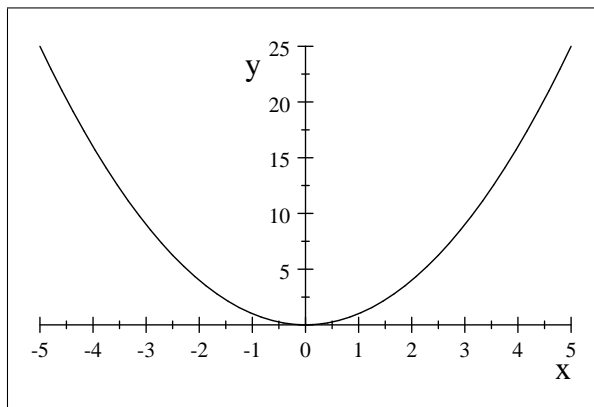
Let me emphasize that

$$f : y \rightarrow x$$

has only one value of  $y$  for each value of  $x$ .

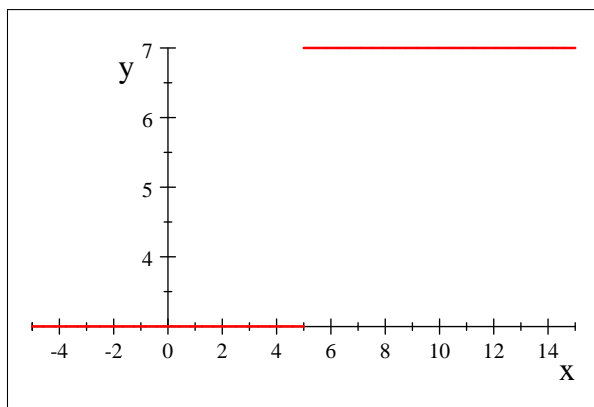
If the mapping is not a unique mapping, it is not a function.

For example  $y = f(x) = x^2$  is a function.



In contrast,

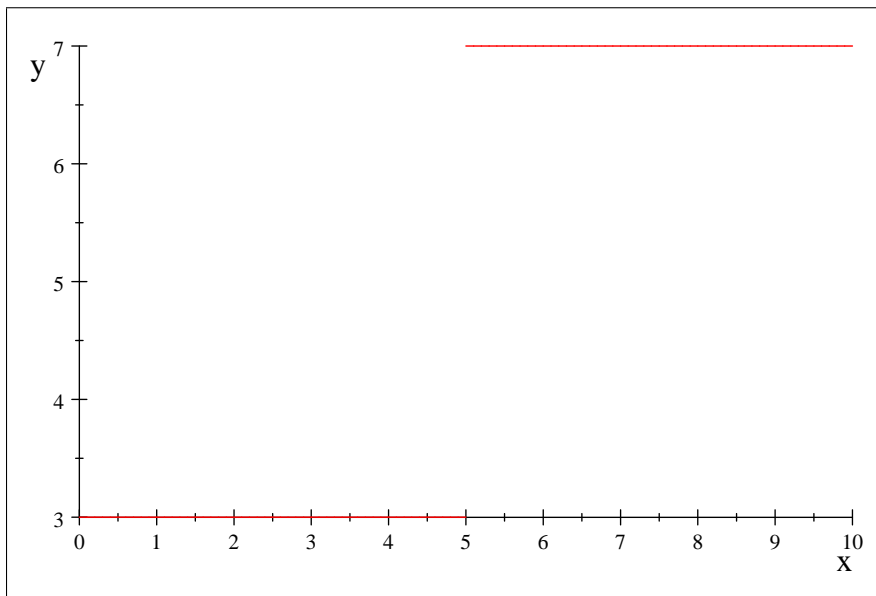
$$y = f(x) = \begin{cases} 3 & \text{if } x \leq 5 \\ 7 & \text{if } x \geq 5 \end{cases} \text{ is not a function}$$



This is an example of a relationship between  $y$  and  $x$ , but  $y$  is not a function of  $x$  because at  $x = 5$ ,  $y$  does not take a unique value, it is either 3 or 7. It would be a function of one replaced  $x \leq 5$  with  $x < 5$  or  $x \geq 5$  with  $x > 5$ .

What if one assumed

$$y = f(x) = \begin{cases} 3 & \text{if } x < 5 \\ 7 & \text{if } x > 5 \end{cases}$$



Is this a function? It seems, so but note that the the function is not defined when  $x = 5$ .

What if the graph is vertical on one or more places?

**1.3 All functions are relationships, but all relationships are not functions. Being a function is sufficient to be a relationship. Being a relationship is necessary but not sufficient to be a function.**

For example

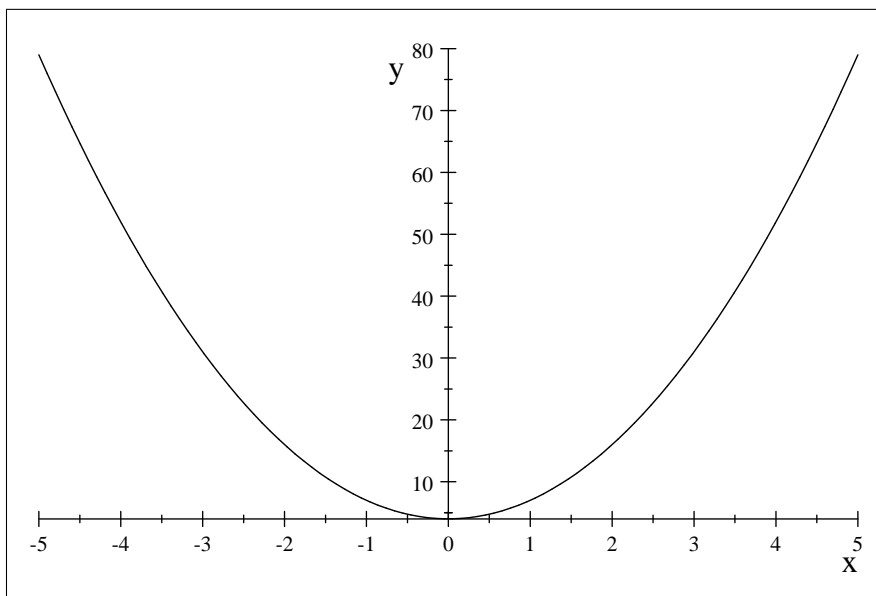
$$\{(x, y) : y \leq x\}$$

is a relationship, but does not define a function  $y = f(x)$ . Why? Graph this relationship with  $x$  on the horizontal axis and  $y$  on the vertical axis. Does this relationship define a function  $x = g(y)$ ?

Does the following relationship define  $y$  as a function of  $x$ ?

$$\{(x, y) : y = a + bx^2\}$$

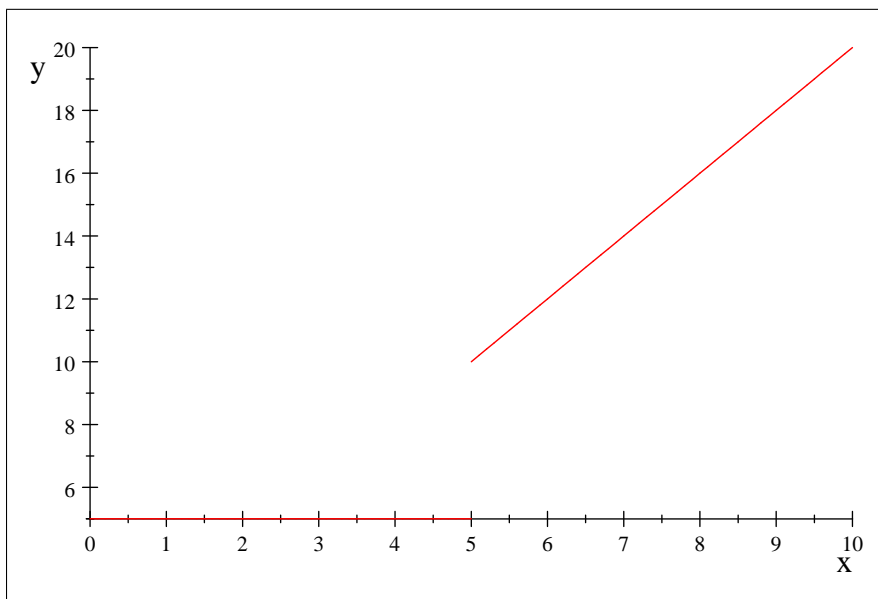
For example, if  $a = 4$  and  $b = 4$



Yes.  $a + bx^2$  defines  $y$  as a function of  $x$ .

Is the following relationship a function?

$$y = f(x) = \begin{cases} 5 & \text{if } x \leq 5 \\ 2x & \text{if } x > 5 \end{cases}$$



Introduce the open and closed circle notation:  $\circ$  and  $\bullet$ .  
 Yes, but note that the mapping from  $y$  to  $x$  is not unique.

Consider some different names for functions

$$y = y(x)$$

$$y = h(x)$$

$$y = fred(x)$$

Consider the following

$$y = f(x) = 3x$$

That is, in this example we have specified what  $f$  means. What then does the following mean?

$$f(6m)$$

Consider the relationship between price and quantity demanded.

**draw a mapping with each price mapping into a range of quantities demanded. Do it with two horizontal lines and a graph with p in the horizontal axis and quantity demanded on the vertical axis?**

#### 1.4 So far in my discussion of functions and relationships, I have implicitly assumed that both $x$ and $y$ are scalars rather than vectors.

What if instead of  $x$  and  $y$  being single variables, they are arrays of variables? If  $x$  represents only one variable,  $x$  is a scalar. If  $x$  represents more than one variable, it is a vector (or matrix). For example, maybe

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

or more generally

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_N \end{bmatrix}$$

Consider the following mapping from points in  $n$ -dimensional  $\mathbf{x}$  space to unique points in one-dimensional  $y$  space. That is

$$y = f(\mathbf{x})$$

where  $y$  is a scalar and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_N \end{bmatrix}$$

This is a function and a relationship.

What about mappings from points in  $n$ -dimensional  $\mathbf{x}$  space to unique points in  $m$ -dimensional  $\mathbf{y}$  space. **Draw a picture** assuming both are in 2-dimensional space.

Is this a relationship? Yes, Is it a function? Yes. It is typically called a *vector-valued* function.

What we think of as a function is a special case of a vector-valued function where  $y$  is a scalar.

In this class, we will deal solely with function where the dependent variable is a scalar.

Give me an example of a mapping from points in  $n$ -dimensional  $\mathbf{x}$  space to points in  $m$ -dimensional  $\mathbf{y}$  space that is a relationship, but not a vector-valued function.

**From points in  $\mathbf{x}$  space to clouds in  $\mathbf{y}$  space.**

What about mappings of clouds of points in  $\mathbf{x}$  space into unique points in  $\mathbf{y}$  space?

For example, Is the following a function?

$$y = f(x) = \begin{cases} 7 & \text{if } x > 5 \\ 1 & \text{if } x \leq 5 \end{cases}$$

Yes. every value of  $x$  maps into one and only one value of  $y$ .

## 1.5 In summary

When I talk or define a function,  $y = f(x)$ , I mean either both  $x$  and  $y$  are scalars, or  $\mathbf{x}$  is a vector and  $y$  is a scalar.

However, keep in mind that there is a more general definition of a function (vector-valued functions)