1 Economic Models and Static Analysis

equilibrium revised February 28, 2012, equilibrium.tex

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2 Equilibrium Defined

2.1 definition

equilibrium is difficult to define

How would you define "equilibrium"?

In the text I used to use for this course, Alfa Chiang, the author, defined equilibrium in the following way

> Equilibrium is a constellation of selected interrelated variables so adjusted to one another so that no inherent tendency to change prevails in the model which they constitute

I don't find this definition very illuminating. Chiang also says

> Equilibrium for a specific model is a situation that is characterized by a lack of tendency to change

This is better, but what he is really talking about is ”static” equilibrium, not equilibrium in general.

Let me begin by explaining how I view the concept of static equilibrium. I will do it in the context of models.

2.1.1

2.1.2 static equilibrium and comparative statics

Variables are things that vary

The variables whose values are determined within our model (the variables we are trying to explain) are called the *endogenous variables*

The variables whose values are determined outside of our model have their values determined *exogenously*, so are called *exogenous variables*. Their values are given from the perspective of our model (models have to start somewhere, that is they start by taking some things as given)

The values of the endogenous variables are related to some of the exogenous variables by behavioral assumptions, which can be described by mathematical functions.

definition: A system is in static equilibrium if, for given values of the exogenous variables, there is no tendency for any of the endogenous variables to change values

> That is, if the endogenous variables are all at their equilibrium values, for the given values of the exogenous variables, then they will remain at those values until the system is perturbed (one or more of the significant exogenous variables change values)

Coke on head

I refer to this type of equilibrium as a static equilibrium because the equilibrium values of the endogenous variables are static (they don't change)

If a system is in equilibrium and one of more of the exogenous variables change, the equilibrium values of the endogenous variables will often change; that is, there will be a new equilibrium.

One of the things that we are often trying to do in economics is to figure out how the equilibrium values of the endogenous variables change when the values of the exogenous variables change

e.g. what happens to the equilibrium level of aggregate income if the exogenous level of government expend increases

The process of doing this is called *comparative statics* because one is comparing two static equilibriums

For example, if we determine that whenever a particular exogenous variable increases in value, the equilibrium value of a particular endogenous variable decreases in value, we have derived a *qualitative comparative static prediction* from our model/theory. It is a *hypothesis* that can be tested.

Now that we understand the notion of a static equilibrium, let's contrast static with dynamic equilibrium.

2.1.3

2.1.4 dynamic equilibrium

In a dynamic equilibrium, the equilibrium values of the endogenous variables do no have to be stationary (static), but rather just have to be on their equilibrium paths.

For example, the planets are in motion around the sun

one can be in dynamic equilibrium while biking or skiing (often we are not)

if for given values of the exogenous variables (or given paths of the exogenous variables), there is no tendency for the paths of the endogenous variables to change, the system is in equilibrium.

2.2 some history

The idea of equilibrium goes back a long way.

A first application was the question of what keeps the earth in place: why does it stay in place rather than plunge through space.

Back around c.700 BC in early Greece there was no sense that the earth was moving, rather, everyone, or at least everyone who thought about it, thought the Earth was motionless in space. A question asked was what kept it in place. The best idea at the time was that Earth sat on something like a cushion or floated on water - Greek gods played a big role in making it work.

The following paraphrases Gottleib 2000, page 11.[1]

Then along came Anaximander (c.610-c.546 BC).[2] Anaximander was a citizen of Miletus (that is, a Milesian), so kind of from the sticks; that is, not an Athenian. He made some wild assumptions. He did not think that the earth needed any sort of cushion - either of water or of anything else - to support it. He held that it rested at the center of a spherical universe, with everything else circling around it, and that it is this pivotal position which explains why it does not fall through space. It is kept in place by equilibrium, as Aristotle explained (describing Anaximander's view, not his own):

> For it behoves that which is established at the center, and is equally related to the extremes, not to be borne one whit more either up or down or to the sides; and it is impossible for it to move simultaneously in opposite directions, so that it stays fixed by necessity.

[1] Anthony Gottlieb,2000,The dream of reason: a history of philosophy from the greeks to the renaissance, W.W. Norton and Company, NY, NY.

[2] All of the old, dead Greek guys seem to have only one name; e.g. Aristotle, Plato, Socrates, etc.

Thus the earth is like Buridan's proverbial ass, which, placed exactly the same distance between two bales of hay could not decide which one to eat and so starved to death in the middle. This idea of Anaximander's is an advanced one in several respects (for our purposes it does matter that it is also quite wrong, not least in its premise that the earth is at the center of things). First of all, it is pleasingly mathematical. Like a trapeze artist who jumps into empty space, confident that his partner will swing over to catch him, Anaximander bravely stepped beyond the realm of material support and trusted in a mathematical idea to catch the earth. Anaximander's principle of equilibrium is invisible and impersonal, and yet it is as powerful as one of the gods. This idea was a bit too novel and the cushion assumption was quickly resurrected.

Now we believe that the earth's path, its orbit, remains static because of the existence of gravity. The assumption that there is something called gravity is also a pretty wild assumption - attraction at a distance. Isaac Newton created the assumption of gravity. To my knowledge, no one as ever proven there is such a thing, things just behave as if it exists.

2.3 the continuum from partial to general equilibrium models

As we make a model more general, variables that were previously exogenous to the model become endogenous (their values are determined within rather than outside of the model)

very partial models explain the values of only a few variables. E.g., very simple Keynesian models that explain only Y and C but take I, G, and T as exogenous

very general models determine the values of many variables

Note that no model can explain everything; that is, every model must have some exogenous variables; it must start somewhere.

2.4 Simple partial equilibrium models

2.4.1 Model 1

Definitions:

$x_1^d \equiv$ the quantity demanded of good 1
$x_1^s \equiv$ the quantity supplied of good 1
$p_i \equiv$ the price of good i, $i = 1, 2, 3$.

Assumptions:
p_2, and p_3 are exogenous
$x_1^d = x_1^d(p_1, p_2, p_3)$
$x_1^s = x_1^s(p_1, p_2, p_3)$

$x_1^d = x_1^s$
What are the endogenous variables? p_1, x_1^d, x_2^s
What will determine the equilibrium values of the endogenous variables? p_2 and p_3

2.4.2 Model 2 (a more general version of Model 1)

Definitions:

$x_i^d \equiv$ the quantity demanded of good i, $i = 1, 2$

$x_i^s \equiv$ the quantity supplied of good i, $i = 1, 2$

$p_i \equiv$ the price of good i, $i = 1, 2, 3$.

Assumptions:

only p_3 is exogenous.

$x_1^d = x_1^d(p_1, p_2, p_3)$

$x_1^s = x_1^s(p_1, p_2, p_3)$

$x_1^d = x_1^s$

$x_2^d = x_2^d(p_1, p_2, p_3)$

$x_2^s = x_2^s(p_1, p_2, p_3)$

$x_2^d = x_2^s$

The Model 2 assumptions can be written more compactly as

$x_i^d = x_i^d(p_1, p_2, p_3)$, $i = 1, 2$.

$x_i^s = x_i^s(p_1, p_2, p_3)$, $i = 1, 2$.

$x_i^d = x_i^s$

In this case, what are the equilibrium quantities of the endogenous variables functions of?

Why might we prefer Model 1 to Model 2?

simpler

depends on what you want to explain

Why might we prefer Model 2 to Model 1?

?

?

2.4.3 Model Three

Assume q_d, and q_s, and p are the endogenous variables and that

$$q_d = a - bp$$
$$q_s = -c + dp$$
$$q_d = q_s$$

$a, b, c,$ and d are exogenous parameters. Note "exogenous parameters" not "exogenous variables"

Further assume that
$(a, b, c, d) > 0$ and $(ad - bc) > 0$

Not worrying about the explicit functional forms, what general form will the solution take; that is, what will be a function of what?

$q_d = q_s = q^e = q^e(a, b, c, d)$
$p^e = p^e(a, b, c, d)$ The endogenous variables as functions of the exogenous parameter values.

Solve for the equilibrium values of the endogenous variables

$p^e = p^e(a, b, c, d) = \frac{(a+c)}{(b+d)}$

Note that, by assumption, $(b + d) > 0$ and $(a + c) > 0$
and
$q^e = q^e(a, b, c, d) = \frac{(ad-bc)}{(b+d)}$

Note that, by assumption, $(ad - bc) > 0$.

Does the three equation model constitute a theory? Definitions? Assumptions? Predictions?

What are some predictions that can be derived from this theory?

$p^e = p^e(a, b, c, d) = \frac{(a+c)}{(b+d)}$

$q^e = q^e(a, b, c, d) = \frac{(ad-bc)}{(b+d)}$

$p^e > 0$

$q^e > 0$

Derive a comparative static prediction from this theory?

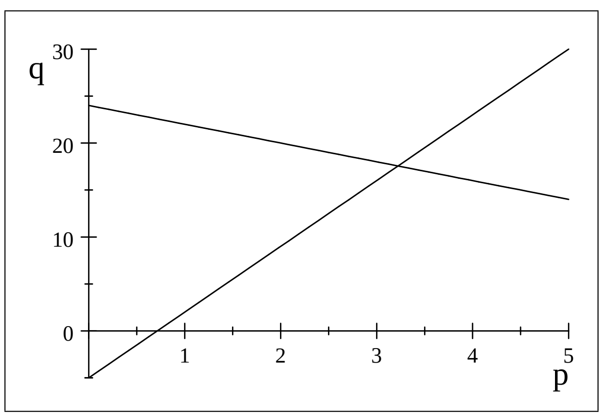
e.g. If, c.p., a increases, p^e will increase. Looking ahead, use a partial derivative to derive a more specific prediction wrt to what will happen to p^e if a increases

What if I replace
$q_d = a - bp$
$q_s = -c + dp$
$q_d = q_s$

with
$q_d = 24 - 2p$

$q_s = -5 + 7p$
$q_d = q_s$

So $24 - 2p = -5 + 7p$, Solution is: $p^e = \frac{29}{9}$. What is q^e? Solve for q^e using either the demand or supply function, with $p^e = \frac{29}{9} = 3.2222$. That is, $q_d^e = 24 - 2p^e = 24 - 2(\frac{29}{9}) = \frac{158}{9} = 17.556 = q^e$



Is this model more or less general than the previous one? Why? Which is better? The more specific the assumptions the more specific the predictions.

2.4.4 Model 4: A simple nonlinear model of supply and demand

$q_d = 4 - p^2$

$q_s = 4p - 1$
$q_d = q_s$

Solve for the equilibrium price and quantity.

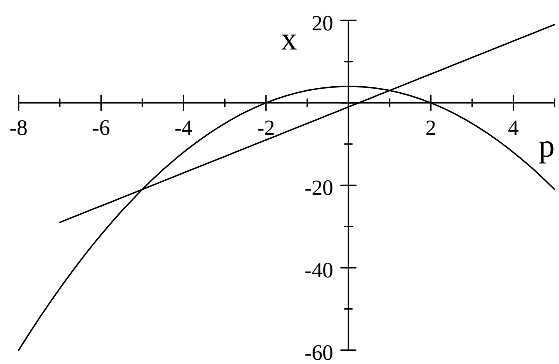
Solve $4 - p^2 = 4p - 1$ for p
This implies
$p^2 + 4p - 5 = 0$, which is a quadratic equation in p

One could solve by factoring or by using the quadratic formula

Quadratic equations are of the form $ax^2 + bx + c = 0$. The solution is of the form $x = \frac{-b \pm (b^2 - 4ac)^{1/2}}{2a}$

How many solutions are there?

$p = 1, -5$. If $p = 1$ then $x = 3$
What do the demand and supply curves look like?



Supply and Demand

Does this makes sense?

Should we add further assumptions to guarantee that there is one solution and that it makes sense? What assumption would you add. If one want to guarentee a postive price and quantity one could or example restrict the demand and supply functions as follows:

$q_d = 4 - p^2$, $p > 0$

$q_s = 4p - 1$, $p > 0$

2.4.5 Model 5: A simple Keynesian macro model

Assume the following simple Keynesian macroeconomic model:

$$Y = C + I^0 + G^0$$

$$C = a + bY,\ a > 0,\ 0 < b < 1$$

$Y \equiv$ national income
$C \equiv$ consumption of domestically produced goods
$G^0 > 0 \equiv$ government expenditures
$I^0 > 0 \equiv$ is the exogenous level of investment

What is the general form of the solution? Solve for the equilibrium levels of income and consumption.

$Y^e = Y(a, b, I^0, G^0) = \frac{a+I^0+G^0}{(1-b)} > 0$, $C^e = C(a, b, I^0, G^0) = a + \frac{b(a+I^0+G^0)}{(1-b)} = \frac{a+b(I^0+G^0)}{(1-b)} > 0$

Why can we predict that $Y^e > 0$ and $C^e > 0$?

What happens to the equilibrium level of income if the marginal propensity to consume, b, increases?

One can eyeball $Y^e = \frac{a+I^0+G^0}{(1-b)}$ and see that if $b \uparrow$ then $(1-b) \downarrow$ but remains positive, so Y^e increases. Or, one could take the partial derivative of Y^e wrt b to determine $\frac{\partial Y^e}{\partial b} = \frac{a+I^0+G^0}{(1-b)^2} > 0$

What happen to the equilibrium level of consumption if the marginal propensity to consume, b, increases?

$C^e = C(a, b, I^0, G^0) = a + \frac{b(a+I^0+G^0)}{(1-b)} = \frac{a+b(I^0+G^0)}{(1-b)}$

This is more difficult to eyeball because b appears in the numerator and denominator. $\frac{\partial C^e}{\partial b} = \frac{(I^0+G^0)(1-b)-(a+b(I^0+G^0))(-1)}{(1-b)^2} = \frac{G^0+a+I^0}{b^2-2b+1} \lesseqgtr 0$????

If in the 1920's President Hoover had believed in this simple Keynesian model, we might have avoided much of the Great Depression.

What would be the equilibrium level of income if the above model were true and econometricians have accurately estimated that $I^0 = 10$, $G^0 = 5$, $a = 20$ and $b = .5$? 70

2.5 Moving towards general equilibrium (G.E.)

Consider the simple partial equilibrium model of supply and demand that we first looked at

Model 1
$x_1^d = x_1^d(p_1, p_2, p_3)$
$x_1^s = x_1^s(p_1, p_2, p_3)$

$x_1^d = x_1^s$
where p_2 and p_3 were assumed exogenous.

Then we generalized the model a bit (made p_2 endogenous) to get Model 2

$x_i^d = x_i^d(p_1, p_2, p_3)$
$x_i^s = x_i^s(p_1, p_2, p_3) \quad i = 1, 2.$

$x_i^d = x_i^s$
Now consider the much more general model

$x_i^d = x_i^d(p_1, p_2, ..., p_N)$
$x_i^s = x_i^s(p_1, p_2, ..., p_N) \quad i = 1, 2, ..., N.$

$x_i^d = x_i^s$
where all prices and quantities are endogenous.

This is an equilibrium model of the supply and demand for all N commodities. It consists of $3N$ equations and $3N$ unknowns, where N is the number of goods in the economy, a really BIG number.

What is the general form of the solution? Equilibrium prices and quantities are going to all be a function of all of the exogenous parameters in all of the supply and demand function.

If you knew the functional forms of the N demand functions and the functional forms of the N supply functions, how would you proceed to find the equilibrium prices and quantities?

The solution is called a general equilibrium because it incorporates all of the interdependencies among the N markets. Does it have a solution? Might it have more than one solution?

This is a very important question in economics. What if a solution did not exist? (see below)

2.5.1 Existence in G.E.

Assume x_i^d, x_i^s, and p_i are all endogenous.

$x_i^d = x_i^d(p_1, p_2, ..., p_N)$

$x_i^s = x_i^s(p_1, p_2, ..., p_N) \quad i = 1, 2, ..., N.$

$x_i^d = x_i^s$

This is an equilibrium model of the supply and demand for N commodities. It consists of $3N$ equations and $3N$ unknowns.

One way to determine whether a solution exists is to find the solution. If you find it, it exists. However, if you followed this path and a solution did not exist, you would wander the world forever doing algebra. (Consider the question "can George get a date". If he went on a date last Saturday, this proves he can get a date.). Proving he can get a date, even though he never has had a date, is a more difficult endeavor.

If it is a complicated system, it would be nice to know that a solution exists before one starts searching for it. It is pointless to search if what you are looking for does not exist, either for the equilibrium or for a date. What is the point of me looking for "Mr. Right" if he does not exist?

System of equations do not always have solutions

Whether a solution exists is called the *existence problem.*

An important question in economics is whether market equilibrium for all goods and services is a theoretical possibility. That is, is there a set of prices that will simultaneously equate supply and demand in all markets?

A lot of the economics you have learned assumed that such an equilibrium exists. That is, most of what you have learned in economics is about equilibriums.

If they can't exist, what you have learned is of no relevance.

Adam Smith assumed it existed.

In the 1950's, Kenneth Arrow and Gerard Debreu proved that under certain conditions a solution exists. This proof won them the "Nobel" prize in economics and Big salaries.

Consider the following simple two market example:

2.5.2 Model 6

$$x_1^d = a_0 + a_1 p_1 + a_2 p_2$$
$$x_1^s = b_0 + b_1 p_1 + b_2 p_2$$
$$x_1^d = x_1^s$$
$$x_2^d = \alpha_0 + \alpha_1 p_1 + \alpha_2 p_2$$
$$x_2^s = \beta_0 + \beta_1 p_1 + \beta_2 p_2$$
$$x_2^d = x_2^s$$

where the two prices and quantities are endogenous. The $\mathbf{a}$, $\mathbf{b}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are exogenous parameters.

How do we proceed? Substitute equations 1 and 2 into equation 3 and solve for zero. Do the same for the last three equations. On obtains

$$(a_0 - b_0) + (a_1 - b_1)p_1 + (a_2 - b_2)p_2 = 0$$

$$(\alpha_0 - \beta_0) + (\alpha_1 - \beta_1)p_1 + (\alpha_2 - \beta_2)p_2 = 0$$

Simplify by
$c_i \equiv a_i - b_i \quad i = 0, 1, 2.$
$\gamma_i \equiv \alpha_i - \beta_i$

Using this simplification

$$c_0 + c_1 p_1 + c_2 p_2 = 0$$

$$\gamma_0 + \gamma_1 p_1 + \gamma_2 p_2 = 0$$

Solve the first equation (or second) for one of the prices in terms of the other price. Then substitute the result into the other equation to get

$$p_1^e = \frac{c_2\gamma_0 - c_0\gamma_2}{c_1\gamma_2 - c_2\gamma_1}$$

One can also show that

$$p_2^e = \frac{c_0\gamma_1 - c_1\gamma_0}{c_1\gamma_2 - c_2\gamma_1}$$

Do we need additional assumptions to guarantee that the equilibrium prices are defined and both positive? What is necessary and sufficient? Two assumptions:
$c_1\gamma_2 \neq c_2\gamma_1$ This assumption is necessary so that the denominators are not zero, but it is not sufficient.

The other required assumption is that $(c_2\gamma_0 - c_0\gamma_2)$, $(c_0\gamma_1 - c_1\gamma_0)$ and $(c_1\gamma_2 - c_2\gamma_1)$ are all of the same sign.

What are the equilibrium quantities?

As you now suspect, solving $3N$ equations for $3N$ unknowns can be a taxing experience. It would be unfortunate to spend hours, days, or weeks on such an endeavor and never succeed.

If would be unfortunate if your endeavor was for not because the system you were trying to solve did not have a solution; that is, if you were searching for something that doesn't exist.

2.5.3 If one has a system of N equations and N unknowns what are the necessary and sufficient conditions for the system to have a solution?

What is a solution? Each endogenous variable as a function of only the exogenous variables and parameters, not a function of other endogenous variables.

could provide here some examples of solutions and non-solutions
It is often said that

> A solution exists if the number of equations equals the number of unknowns

This is wrong, as I will show in a moment.

> Put simply, for a system to have a solution, one needs a separate "piece" of information for each endogenous variable in the system and those pieces of information must be consistent with one another

More specifically,

> If one has N endogenous variables, then the necessary and sufficient condition for a solution (equilibrium) to exist is that there are N consistent and functionally independent equations in the system

,

Consider the following two examples.

$q^d = 4 - 2p$
$q^s = 1 + p$
$q^d = q^s$

and
$q^d = 4 - 2p$
$q^s = 5 + p$
$q^d = q^s$

What is the solution to each system? For the first,
$4 - 2p = 1 + p$, Solution is: $p = 1$
For the second

$4 - 2p = 5 + p$, Solution is: $p = -\frac{1}{3}$

Both have solutions, but we might not think the solution to the second model is a good economic solution.

So in the above two models, there are three equations and three unknowns, and each has a solution.

However, having the number of equations equal to the number of unknowns does not guarantee that a solution exists.

Ask them to solve the following system for the three endogenous variables.

For example, consider the following system of 3 equations and three unknowns

$$x_1 = a + bp_1$$

$$x_2 = c + dp_1$$

$$x_1 = a + c + (d + b)p_1 - x_2$$

Does this system have a solution? No. Note that equations 1 and 2 imply equation 3; that is, there is no new information in equation 3. Equation 3 is just equation 1 with equation 2 both added and subtracted from the right hand side. The three equations are not functionally independent.

This example demonstrates that when there are three endogenous variables, three equations is not sufficient to generate a solution.

Consider the following system:

$$C = a + bY$$

$$C = c + bY$$

where a, b, and c are exogenous and $a \neq c$. In this case there is no solution. The two equations are inconsistent with one another

The first and second examples sho that number of equations equal to number of unknowns is not sufficient to guarentee a solution.

Is having the number of equations equal the number of unknowns necessary for a solution to exist? Consider the following system

$$C = a + bY$$

$$2C = 2a + 2bY$$

$$C + I^0 = Y$$

This system has three equations and two unknowns, but does have a solution; number of equations equal to number of unknowns is not necessary for a solution to exist. In this trivial example, two of the equations contain the exact same information.

This last example show that number of equation equal to number of unknowns is not even necessary.

With less trivial systems, it is often very difficult to tell is a system of equations has a solution. One needs to determine the number of independent pieces of information, and then make sure those independent pieces are consistent with one another.

- If all the equations are linear, we can check this with a mathematical construction called *determinants*.
- If some or all of the equations are nonlinear, the problem is much tougher.