1 Elasticities

elasticity.tex April 5, 2012

One of the problems with marginal product functions is that the marginal product is sensitive to the units in which x, l, and k are measured. For example, changing the ouput from pounds of stuff to tons of stuff will drastically change the numerical values of the marginal products. It would be nice to be able to denote the productivity of a marginal unit of input in a way that was not sensitive to the units that the inputs and output are measured.

Can we come up with such a measure? Yes

Consider the production function $x = f(k, l)$

1.1 Consider expressing changes in percentage terms.

E..g.

$$\frac{\%\triangle x}{\%\triangle l}$$

is the percentage in output x given a one percent change in the input l, holding k constant. We call this the *elasticity of output with respect to labor*.

Note that

$$\frac{\triangle x}{x} = \%\triangle x \text{ and } \frac{\triangle l}{l} = \%\triangle l$$

Therefore

$$\frac{\%\triangle x}{\%\triangle l} = \frac{\triangle x/x}{\triangle l/l} = \frac{\triangle x}{\triangle l}\frac{l}{x}$$

Definition: The point elasticity of x with respect to l is

$$\frac{\%\triangle x}{\%\triangle l} = \lim_{\triangle l \to 0} \frac{\triangle x}{\triangle l}\frac{l}{x} = \frac{l}{x}\lim_{\triangle l \to 0} \frac{\triangle x}{\triangle l} = \frac{l}{x}\frac{\partial x}{\partial l}$$

Or more generally, the point elasticity of m with respect to x_i where $m = m(x_1, x_2, ...x_n)$ is

$$\frac{\%\triangle m}{\%\triangle x_i} = \frac{\partial m}{\partial x_i}\frac{x_i}{m}$$

What is $\frac{\%\triangle x}{\%\triangle l}$ for the Cobb-Douglas with constant returns to scale?

$$\frac{\%\triangle x}{\%\triangle l} = \frac{l}{x}\frac{\partial x}{\partial l} = \frac{l}{x}a\alpha l^{\alpha-1}k^{1-\alpha} = \frac{a\alpha l l^{\alpha-1}k^{1-\alpha}}{x} = \frac{a\alpha l^{\alpha}k^{1-\alpha}}{x} = \frac{a\alpha l^{\alpha}k^{1-\alpha}}{al^{\alpha}k^{1-\alpha}} = \alpha$$

Every time the quantity of labor increases by one percent, output increase by $\alpha\%$. It is a constant, independent of the level of l. This would not generally be true, but it is true for the Cobb-Douglas.

Analogously

$$\frac{\%\triangle x}{\%\triangle k} = \frac{k}{x}\frac{\partial x}{\partial k} = \frac{k}{x}a(1-\alpha)l^{\alpha}k^{-\alpha} = \frac{a(1-\alpha)l^{\alpha}kk^{-\alpha}}{x} = \frac{a(1-\alpha)l^{\alpha}k^{1-\alpha}}{x} = \frac{a(1-\alpha)l^{\alpha}k^{1-\alpha}}{al^{\alpha}k^{1-\alpha}} = (1-\alpha)$$

That these two elasticities are constant and one elasticity is one minus the other is quite restrictive. A very simple form for the production function implies very restrictive relationships between the inputs and the output.

Now let me give you an alternative way to calculate elasticities. It is the case that

$$\frac{\%\triangle x}{\%\triangle l} = \frac{l}{x}\frac{\partial x}{\partial l} \equiv \frac{\partial \ln x}{\partial \ln l}$$

I will "prove" (skip it) this last equality in a moment but first let's use it to find the two Cobb-Douglas output elasticities. Given that

$$x = f(k,l) = al^{\alpha}k^{1-\alpha}$$

$$\ln x = \ln a + \alpha \ln l + (1-\alpha)\ln k$$

So

$$\frac{\%\triangle x}{\%\triangle l} = \frac{\partial \ln x}{\partial \ln l} = \alpha$$

and

$$\frac{\%\triangle x}{\%\triangle k} = \frac{\partial \ln x}{\partial \ln k} = (1-\alpha)$$

For the Cobb-Douglas, the ln form of the elasticity formula is the easiest way to proceed. Depending on the functional form, it might or might not be the way to go. It depends on how the function simplifies when one logs both sides of it.

1.1.1

1.1.2 Digression on why $\frac{l}{x}\frac{\partial x}{\partial l} = \frac{\partial \ln x}{\partial \ln l}$

By the chain rule

$$\frac{\partial \ln x}{\partial \ln l} = \frac{\partial \ln x}{\partial x}\frac{\partial x}{\partial l}\frac{\partial l}{\partial \ln l} = \frac{1}{x}\frac{\partial x}{\partial l}\frac{\partial l}{\partial \ln l}$$

Now digress further to determine $\frac{\partial l}{\partial \ln l}$

Consider the function

$$m = \ln l$$

In which case

$$\frac{\partial m}{\partial l} = \frac{\partial \ln l}{\partial l} = \frac{1}{l}$$

What then is

$$\frac{\partial l}{\partial m} = \frac{\partial l}{\partial \ln l} \text{ ?}$$

Start with $m = \ln l$. Invert it to obtain

$$l = e^m$$

Note that

$$\frac{\partial l}{\partial m} = e^m = e^{\ln l} = l$$

Since

$$\frac{\partial l}{\partial m} = \frac{\partial l}{\partial \ln l}$$

$$\frac{\partial l}{\partial \ln l} = l$$

This should not surprise us since

$$\frac{\partial \ln l}{\partial l} = \frac{1}{l}$$

Substituting $\frac{\partial l}{\partial \ln l} = l$ into

$$\frac{\partial \ln x}{\partial \ln l} = \frac{1}{x}\frac{\partial x}{\partial l}\frac{\partial l}{\partial \ln l}$$

One obtains

$$\frac{\partial \ln x}{\partial \ln l} = \frac{1}{x}\frac{\partial x}{\partial l}\frac{l}{1} = \frac{\partial x}{\partial l}\frac{l}{x} \equiv \frac{\%\triangle x}{\%\triangle l}$$

qed

1.1.3 End of digression

Obviously, one can calculate many sorts of interesting elasticities.

For example, consider some cost function

$$c(x, w, r)$$

One might want to know how much min costs will increase in percentage terms if the price of capital increases by one percent.

$$\frac{\%\triangle c}{\%\triangle r} = \frac{\partial c(x, w, r)}{\partial r} \frac{r}{c(x, w, r)} = \frac{\partial \ln c(x, w, r)}{\partial \ln r}$$

For example, there is a fancy cost function called the *Translog.* It is specified in log form[1]

$$\ln c(x, w, r) = a_0 + \ln x + a_1 \ln w + (1 - a_1) \ln r + \frac{1}{2} b_{11} (\ln w)^2 + \frac{1}{2} b_{22} (\ln r)^2 + \frac{1}{2} b_{12} (\ln w)(\ln r)$$

where $a_i > 0 \ \forall i$ and $b_{ij} > 0 \ \forall \ i$ and j.

If the Translog is an accurage representation of a firm's cost function, how much will costs increase in percentage terms if r increases by one percent, holding x and w constant. If x increases by one percent holding w and r constant?

$$\frac{\%\triangle c}{\%\triangle x} = \frac{\partial \ln c(x, w, r)}{\partial \ln x} = 1$$

and

$$\frac{\%\triangle c}{\%\triangle r} = \frac{\partial \ln c(x, w, r)}{\partial \ln r} = (1 - a_1) + b_{22}(\ln r) + \frac{1}{2} b_{12}(\ln w)$$

Practice quiz on elasticities:Assume

$$Y = C + I_0 + G$$

$$C = a + b(Y - T_0), \ a > 0 \text{ and } 0 < b < 1$$

$$G = gY \quad 0 < g < 1 \text{ and } (b + g) < 1$$

Determine what happens to the equilibrium level of income if g increases by 1%.

It can be shown that

$$Y^* = \frac{I_0 + a - bT_0}{1 - b - g}$$

$$C^* = \frac{bI_0 + (1 - g)(a - bT_0)}{1 - b - g}$$

[1] It is never represented in non-log form.

$$G^* = \frac{gI_0 + g(a - bT_0)}{1 - b - g}$$

We know that $\frac{\% \triangle Y^*}{\% \triangle g} = \frac{\partial Y^*}{\partial g} \frac{g}{Y^*}$
Since

$$\frac{\partial Y^*}{\partial g} = \frac{I_0 + a - bT_0}{(1 - b - g)^2}$$

$$\begin{aligned}
\frac{\% \triangle Y^*}{\% \triangle g} &= \frac{\partial Y^*}{\partial g} \frac{g}{Y^*} \\
&= \frac{(I_0 + a - bT_0)g}{(1 - b - g)^2 Y^*} \\
&= \frac{(I_0 + a - bT_0)g}{(1 - b - g)^2 \frac{(I_0 + a - bT_0)}{(1 - b - g)}} \\
&= \frac{(I_0 + a - bT_0)g(1 - b - g)}{(1 - b - g)^2 (I_0 + a - bT_0)} \\
&= \frac{g}{(1 - b - g)}
\end{aligned}$$

So, in this model, if g, the marginal propensity to spend on the part of the government, increase by 1%, equilibrium income will increase by $\frac{g}{(1-b-g)}$ percent.