

# 1 Economic Application of Derivatives

deriv-applic.tex and .pdf April 5, 2007

In earlier notes, we have already considered marginal cost as the derivative of the cost function. That is

$$mc(x) = c'(x)$$

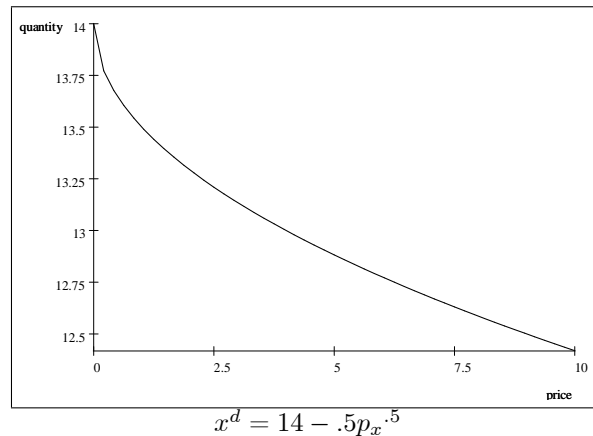
How would you define, in words, the cost function and the marginal cost function?

## 1.1 Now let's consider total revenue, average revenue and marginal revenue

Start by defining the demand function and inverse demand function for some product  $x$ .

Define the demand function for product  $x$  as  $x = x(p_x)$ , where  $p_x$  is the price of product  $x$ . The function  $x = x(p_x)$  identifies the number of units of  $x$  that will be purchased (demanded) as a function of  $p_x$ .

Draw a demand curve with quantity on the vertical axis. Note what is on each axis.

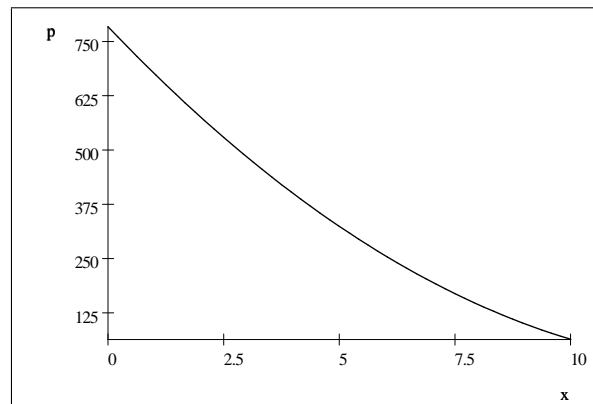


Typically when we draw "demand" curves on blackboards we are really drawing inverse demand functions.

Now consider the inverse of the demand function  $p_x = p_x(x)$ . That is, price as a function of quantity. This called the inverse demand function.

What is the inverse demand function corresponding to  $x^d = 14 - .5p_x^5$ , Solve for  $p_x$ . Solution is:  $\{p_x = 4.0 (x^d)^2 - 112.0x^d + 784.0\}$ ? where  $4.0 (x^d)^2 - 112.0x^d + 784.0 = 4.0 (x^d - 14.0)^2 = 4.0 (14 - x^d)^2$

Graph it.



$$p_x = 4.0 (x^d)^2 - 112.0x^d + 784.0$$

In words, how would you define the inverse demand function  $p_x = p_x(x)$ ?

If  $p_x = p_x(x)$ , define the total revenue function as a function of  $x$ .  $tr = g(x)$ . Remember that total revenue is price multiplied by quantity.

$$tr(x) = g(x) = p_x(x)x$$

Can you express total revenue as a function of price?  $tr(p_x) = h(p_x) = p_x x(p_x)$

That is, one can define total revenue in terms of either  $x$  or  $p_x$ . Note they have different functional forms,  $g$  and  $h$ .

### 1.1.1 Total revenue as a function of $x$

$$tr = g(x) = p_x(x)x$$

Note that average revenue per unit of product sold is total revenue, in terms of quantity, divided by quantity, so

$$ar(x) = \frac{g(x)}{x} = \frac{p_x(x)x}{x} = p_x(x)$$

That is, average revenue equals price.

Find marginal revenue in terms of  $x$ ,  $mr(x)$ . Marginal revenue is the instantaneous change in revenue.

$$mr(x) = \frac{dtr(x)}{dx} = \frac{d(g(x))}{dx} = \frac{d(p_x(x)x)}{dx} = p_x(x) + \frac{dp_x(x)}{dx}x$$

That is, the marginal revenue associated with a change in the quantity sold is the price of  $x$  plus the derivative of the price wrt  $x$  multiplied by  $x$ .

Does this make sense to you? When  $x$  increases by, for example, one unit, one takes in extra revenue from the sale of that last unit,  $p_x(x)$ , but to sell that last unit price had to decline, so revenue earned on all previous units has changed by  $\frac{dp_x(x)}{dx}x$ , so the total effect is the sum of these two separate effects.

Find  $mr(x) - ar(x)$

$$mr(x) - ar(x) = \left[ p_x(x) + \frac{dp_x(x)}{dx}x \right] - p_x(x) = \frac{dp_x(x)}{dx}x$$

Can the difference between marginal and average revenue ever be positive?

- Yes - but only if the demand curve slopes up.
- What is the difference between marginal revenue and average revenue from the perspective of a **competitive** firm?

From the perspective of a competitive firm  $p_x$  is a constant, so  $\frac{dp_x}{dx} = 0$ . The competitive firm can sell as much as it wants at the competitive price. So, for the competitive firm

$$mr(x) - ar(x) = \frac{dp_x(x)}{dx}x = 0$$

- Therefore, in pure competition,  $p_x = ar = mr$ .

- What if the firm is not competitive. That is, what if from the firm's perspective  $\frac{dp_x(x)}{dx} < 0$ . To sell more the firm needs to lower its price. In this case,

$$mr(x) - ar(x) = \frac{dp_x(x)}{dx}x < 0$$

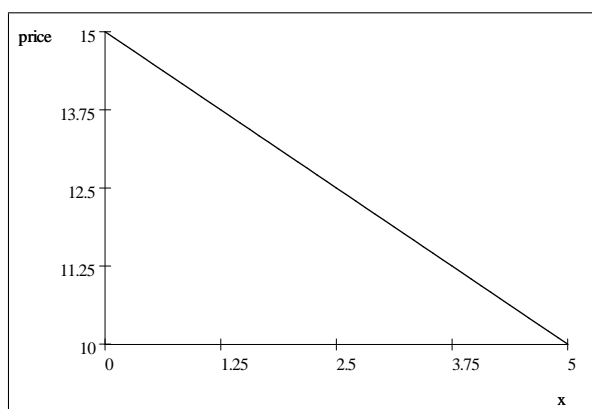
- That is, for the noncompetitive firm,  $p_x(x) = ar(x) > mr(x)$
- Draw graphs and identify the vertical distance between the two curves.

Consider the following problem.

## 1.2 You own a movie theatre.

Your costs are some constant,  $m$ , independent of the number of tickets you sell. Further assume that the inverse demand function that you face is

$$p = p(x) = 15 - x, \quad x \geq 0$$



Find your marginal revenue function and then use it to determine what ticket price you should charge to maximize your profits.

Need to derive the total revenue function. In terms of  $x$  or  $p$ ? In terms of  $p$  because  $p$  is the choice variable.

If  $p = p(x) = 15 - x$ ,  $x(p) = 15 - p$ . So total revenue as a function of  $p$  is  $tr(p) = (15 - p)p = 15p - p^2$ .

Note that I could have determined total revenue as a function of  $x$  rather than  $p$  but since  $p$  is the choice variable is seemed more straightforward to determine it as a function of  $p$ .

So, profit as a function of  $p$  is

$$\pi(p) = 15p - p^2 - m$$

Note that in this case

$$\frac{d\pi(p)}{dp} = \frac{dtr(p)}{dp} = mr(p) = 15 - 2p$$

Does knowledge of the marginal revenue function help us to determine what price to set? Yes.

When is  $mr(p) > 0$ ,  $mr(p) < 0$ ,  $mr(p) = 0$ ? Find that  $p$ ,  $p^*$ , such that  $mr(p^*) = 0$ . That is,

$$\text{solve } 15 - 2p = 0 \text{ for } p$$

$$p^* = 7.5$$

Should the movie theatre charge this price for tickets? If at the current price  $mr(p) > 0$ , the firm can increase its total revenue by increasing  $p$ . If at the current price  $mr(p) < 0$ , the firm can increase its total revenue by decreasing  $p$ . This suggest that total revenue from the theatre will be maximized when  $mr(p^*) = 0$ .

Will profits be maximized when total revenues are maximized? Is this typically the case?

Will the movie theatre make a positive profit? Yes if  $m < 56.25$ . How did you determine this? At a ticket price of \$7.50 how many tickets will be sold? 7.5 tickets.  $7.5 \cdot (7.5) = 56.25$  will be total revenue. What should it do if  $m = \$60.00$ ?

What price should the movie theatre charge if costs are \$60.00 whenever the theatre has one or more paying patrons, and zero otherwise? A price so high that no one will buy a ticket. Demand will be zero at any price  $\geq \$15.00$ . Why?

What have we just done? We have solved for a firm's profit maximizing price and output under two different assumptions about the firm's costs (fixed costs and variable costs). When we assumed costs were variable we did not assume they were very variable.

What I have been doing is introducing the "Theory of the Firm". You should review the chapters in your micro text on the theory of the firm. That is, all the stuff on production functions, cost functions, revenue, profit maximization, etc.

**1.2.1 Continuing with our movie theatre problem. Make the problem a little more interesting.**

Assume

$$c = c(x) = m + x^2$$

We want to solve for either that  $p$  or that  $x$  that maximizes the movie theatre's profits. Will it be easier to set the problem up as one of finding  $x$  or finding  $p$ ? The demand function is expressed in terms of  $x$  as a function of  $p$ , and the cost function is expressed as a function of  $x$ . We need to express profits as a function of one or the other, but not both. Let's express total revenue as a function of  $x$ , since cost is already expressed as a function of  $x$ .

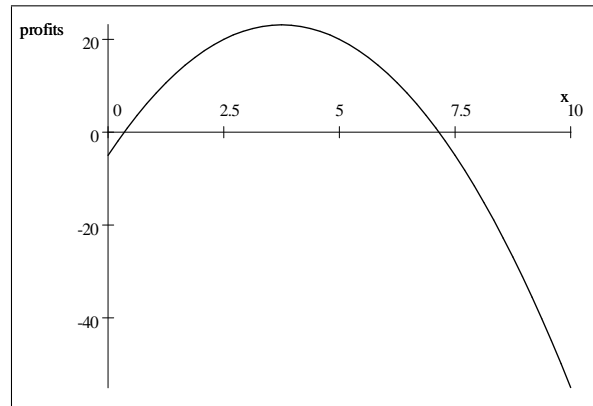
$$p(x) = 15 - x$$

$$\text{So, } tr(x) = 15x - x^2$$

So profits, in this case, as a function of  $x$  are

$$\pi(x) = 15x - x^2 - c(x) = 15x - x^2 - m - x^2 = 15x - m - 2x^2$$

If, for example,  $m = 5$ ,



What output level (and price) should the movie theatre choose to maximize its profits.

If

$\pi'(x) > 0$  the firm should increase output (lower price)

$\pi'(x) < 0$  the firm should decrease output (increase price)

This suggests that profit's might be maximized at that  $x$ ,  $x^*$ , such that  $\pi'(x^*) = 0$ . Find this  $x$  and check to see if profits are maximized at this point. First determine that

$$\pi'(x) = 15 - 4x = 0$$

implies that  $x^* = 3.75$ . Notice that at this level of output,  $mr(3.75) = mc(3.75) = 7.5$  so  $\pi'(3.75) = 0$ .

What price would it charge to sell 3.75 tickets? Plug 3.75 into the inverse demand function to get  $p^* = \$11.25$

What are the firm's profits, total revenues, total costs?

### 1.2.2 Consider a different movie theatre problem:

Same cost function but now assume the movie theatre is a competitive firm so can sell as many tickets as it wants at the exogenous price  $p$ . Further assume the theatre is large.

In this case,

$$\pi(x) = px - c(x) = px - m - x^2$$

In this case

$$\pi'(x) = p - mc(x) = p - 2x$$

Marginal profits as a function of  $x$  are zero when  $x = x(p) = p/2$ . Is this the profit maximizing output level? Is this the firm's supply function. Note that the supply function for a competitive firm identifies the profit maximizing output level as a function of the exogenous competitive price.