1 Economic applications of total differentials

April 26, 2012

Total differentials are very useful for getting a better understanding of a lot of the common economic graphs. Graphs such as isoquants, indifference curves, and isocost lines.

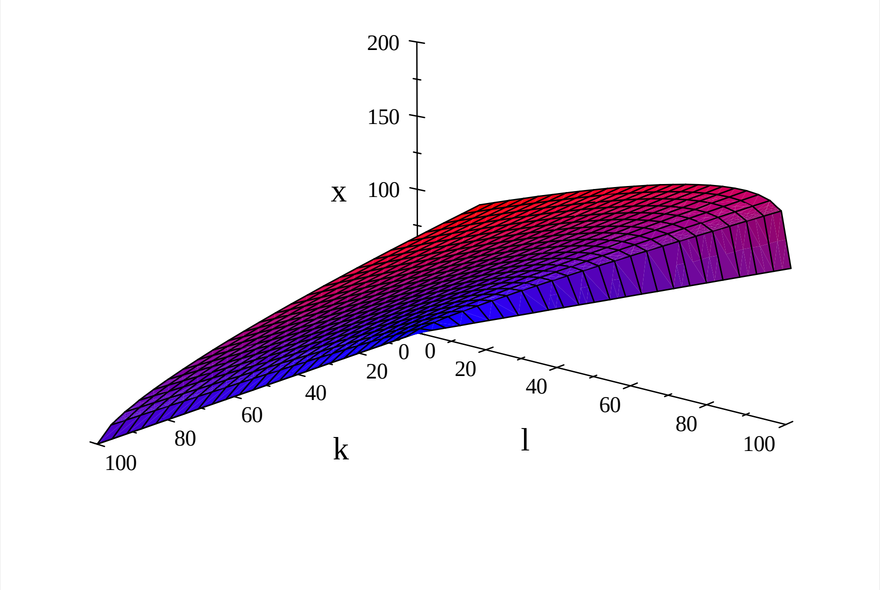
Let's start with production functions and isoquants.

Consider the production function

$$y = f(k, l)$$

Visualize it in three-dimensional space

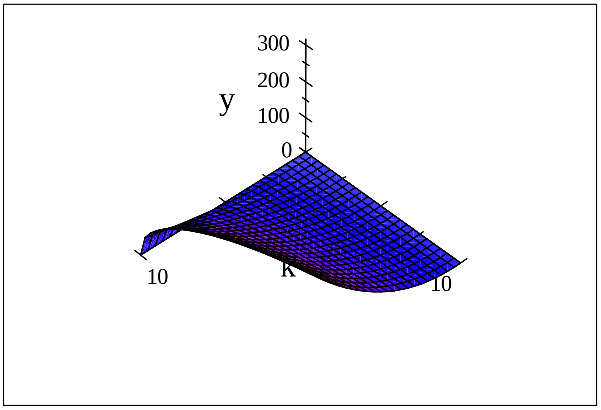
It might look like



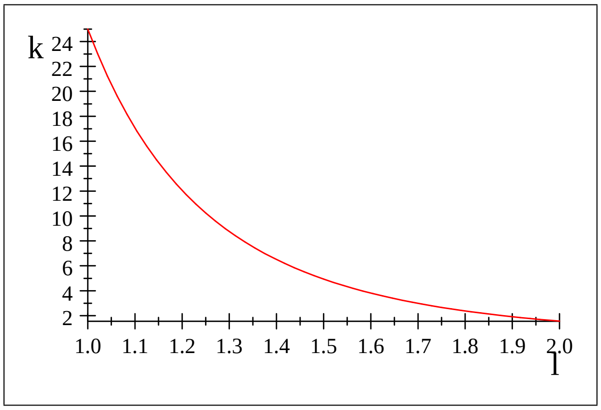
1.1 Now define a isoquant

> **definition:** The isoquant for the output level, y^0, identifes all those combinations of k and l that are just capable of producing y^0 units of output.

Picture a production function, $y = l^2 k^{.5}$, and the the associated isoquant for $y = 5$



On can derive the isoqant for $y = 5$ by solving $5 = l^2 k^{.5}$ for k. The solution is $k = \frac{25.0}{l^4}$. The next graph is the corresponding isoquant for $y = 5$



Note that the isoquants are the boundaries of input requirement sets. The input requirement set for producing y^0 units of output is all combinations of k and l that are capable of producting y^0. That is

$$I(y^0) \equiv \left\{(k,l) : f(k,l) \geq y^0,\, l \geq 0,\, k \geq 0\right\}$$

Picture the input requirement set on both the floor and board.

1.2 Note that as one moves along an isoquant, $dy = 0$

Since $dy = 0$ along the isoquant, the concept of a total differential is a second useful method for deriving the slope of the isoquant from a production function. If $y = f(k, l)$

$$dy = f_k dk + f_l dl$$

But along the isoqant $dy = 0$, so along the isoquant

$$0 = f_k dk + f_l dl$$

Therefore, as one moves along the isoquant

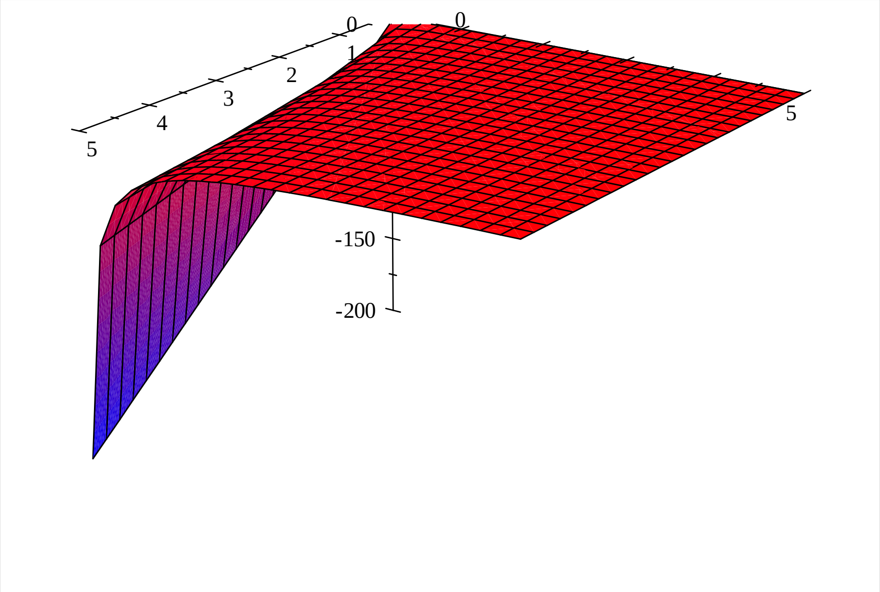
$$f_k dk = -f_l dl$$

Rearranging

$$\left.\frac{dk}{dl}\right|_{dy=0} = -\frac{f_l}{f_k} = -\frac{mp_l(k,l)}{mp_k(k,l)} = -MRTS_{lk}$$

$MRTS_{lk}$ is the marginal rate of technical substitution of labor for capital. That is, the rate as which one can substitue capital for labor holding output constant. $\left.\frac{dk}{dl}\right|_{dy=0}$ is the slope of the isoquant, so $MRTS_{lk}$ is the negative of the slope of the isoquant. Said loosely, $\left.\frac{dk}{dl}\right|_{dy=0}$ is how much one can reduce k when l is increased by one unit.

Consider again the production function $y = l^2 k^{.5}$. The graph of its isoquant for $y = 5$ is above. So, for this production function $\left.\frac{dk}{dl}\right|_{dy=0} = -\frac{f_l}{f_k} = -\frac{2lk^{.5}}{.5l^2k^{-.5}} = -4.0\frac{k}{l}$

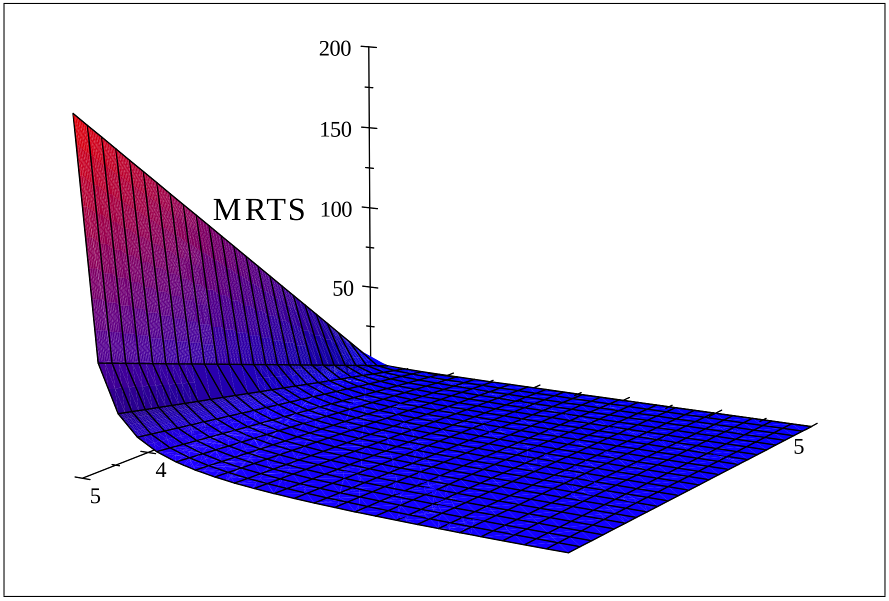


$$\left.\frac{dk}{dl}\right|_{dy=0} = -4.0\frac{k}{l} \text{ for } y = l^2k^{.5}$$

So, what is this showing? It is plotting the slope of the relevant isoquant at each combination of labor and capital. It shows how much k needs to be reduced when l is increased if the intent is to hold y constant, as a function of the current levels of k and l.

It is always negative because when l is increased, k must decrease. How much, a little or a lot, depends on the amounts of k and l currently in use.

If we graph the negative of $\left.\frac{dk}{dl}\right|_{dy=0}$, we have a graph of the $MRTS_{lk}$; that is it is plotting the marginal rate of technical substitution of labor for capital at each labor capital combination.



1.3 Consider the general form of the Cobb-Douglass production function $x = f(k,l) = Al^{\alpha}k^{\beta} \quad A > 0$, $\alpha > 0$, $\beta > 0$

Given that it is always the case that

$$\left.\frac{dk}{dl}\right|_{dy=0} = -\frac{f_l}{f_k} = -\frac{mp_l(k,l)}{mp_k(k,l)}$$

One just need to find the $mp_l(k,l)$ and $mp_k(k,l)$ for the CD; then divide and simplify

$$f_1 = mp_l(k,l) = \alpha A l^{\alpha-1}k^{\beta}$$

and

$$f_k = mp_k(k,l) = \beta A l^{\alpha}k^{\beta-1}$$

So

$$\left.\frac{dk}{dl}\right|_{dx=0} = -\frac{mp_l(k,l)}{mp_k(k,l)} = -\frac{\alpha A l^{\alpha-1}k^{\beta}}{\beta A l^{\alpha}k^{\beta-1}} = -\frac{\alpha A l^{\alpha}l^{-1}k^{\beta}}{\beta A l^{\alpha}k^{\beta}k^{-1}} = -\frac{\alpha l^{-1}}{\beta k^{-1}} = -\frac{\alpha k}{\beta l} < 0$$

That is, the slope of the isoquant for this Cobb-Douglas production function at the point (k^0, l^0) is $-\frac{\alpha k^0}{\beta l^0}$ and the $MRTS_{lk} = \frac{\alpha k^0}{\beta l^0}$. Put simply, increasing labor by one unit will allow the producer to decrease capital by $\frac{\alpha k^0}{\beta l^0}$ units.

What have we determined about the slope of this isoquant? By assumption, it has a negative slope for positive values of k and l.

For the C.D. is it a straight line? No. How do we know this. The slope is a function of k and l.

What happens to the slope of the isoquant as l increases? To answer this take the partial derivative of the slope with respect to l.

$$\frac{\partial(\left.\frac{dk}{dl}\right|_{dx=0})}{\partial l} = \frac{\partial(-\frac{\alpha k}{\beta l})}{\partial l} = \frac{\partial(-(\frac{\alpha}{\beta})kl^{-1})}{\partial l} = (\frac{\alpha}{\beta})kl^{-2} > 0$$

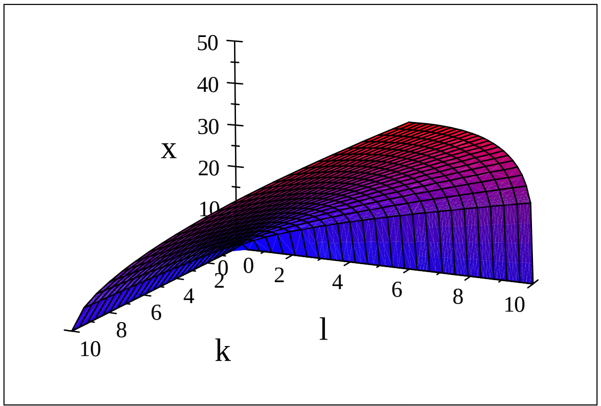
The slope of the isoqant becomes flatter as l increases.

1.3.1 Let's do an numerical example.

Assume

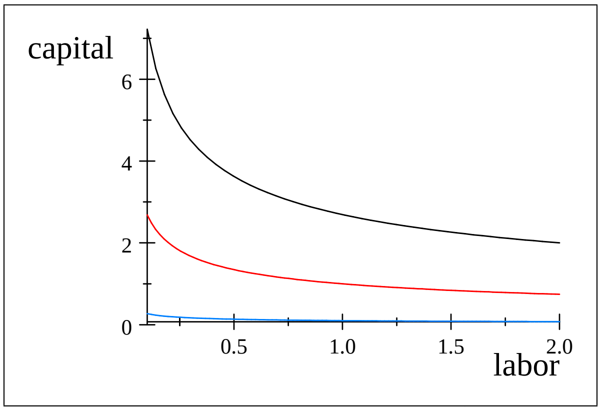
$$x = f(k, l) = 5l^{.3}k^{.7}$$

It looks like



$x = f(k, l) = 5l^{.3}k^{.7}$

And the isoquants look like



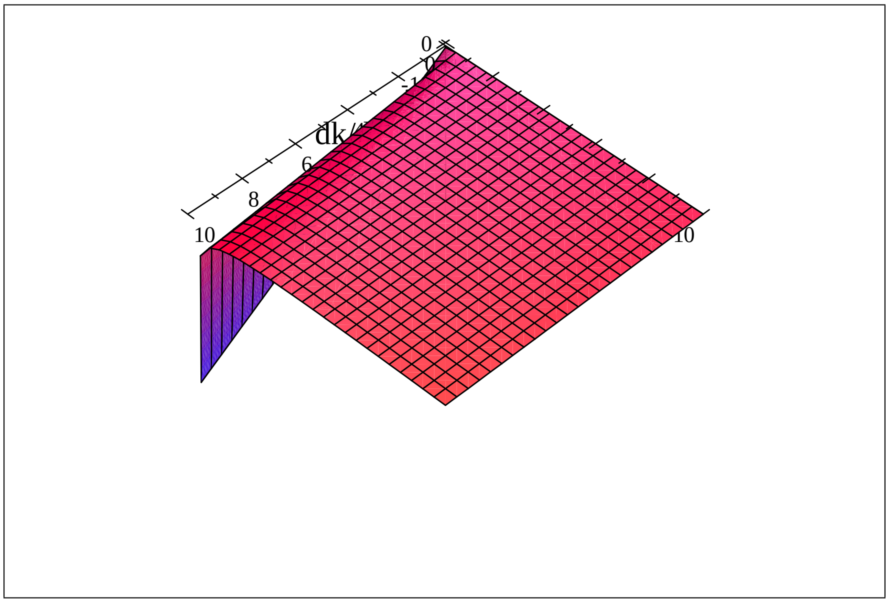
some isoquants for $x = 5l^{.3}k^{.7}$

We know that for the Cobb-Douglas in general

$$\left.\frac{dk}{dl}\right|_{dx=0} = -\frac{\alpha k}{\beta l}$$

So

$$\left.\frac{dk}{dl}\right|_{dx=0} = -\frac{.3k}{.7l}$$



$\frac{dk}{dl}\Big|_{dx=0} = -\frac{.3k}{.7l}$

If $k^0 = 10$ and $l^0 = 1$ (a very capital-intensive combination of l and k), one unit of labor substitutes for how many units of capital.

$$MRTS_{lk}(k^0, l^0) = MRTS_{lk}(5, 2) = \frac{.3(10)}{.7(1)} = 4.29$$

At this point, one unit of labor substitues for 4.29 units of capital (if labor is increased by one unit, capital can be decreased by 4.29 units.). At this point the isoquant is steep.

Now consider the $MRTS$ at $k^1 = 2$ and $l_1 = 5$, $MRTS_{lk}(k^1, l^1) = MRTS_{lk}(2, 5) = \frac{.3(2)}{.7(5)} = .171$. That is, at this more labor intensive combination of labor and capital, one unit of labor substitutes for only .171 units of capital. At this point

the isoquant is relatively flat.

1.4 Consider now the more general CES production function

$$x = f(k, l) = [\alpha_0 + \alpha l^\rho + \beta k^\rho]^{1/\rho}$$

What do I mean by "more general"? More general in what sense?

In this case,

$$\frac{\partial x}{\partial l} = mp_l(k,l) = (1/\rho)\left[\alpha_0 + \alpha l^\rho + \beta k^\rho\right]^{(1/\rho)-1} \rho\alpha l^{\rho-1}$$

$$= \left[\alpha_0 + \alpha l^\rho + \beta k^\rho\right]^{(1/\rho)-1} \alpha l^{\rho-1}$$

and

$$\frac{\partial x}{\partial k} = mp_k(k,l) = (1/\rho)\left[\alpha_0 + \alpha l^\rho + \beta k^\rho\right]^{(1/\rho)-1} \rho\beta k^{\rho-1}$$

$$= \left[\alpha_0 + \alpha l^\rho + \beta k^\rho\right]^{(1/\rho)-1} \beta k^{\rho-1}$$

So,

$$MRTS_{lk} = \frac{\left[\alpha_0 + \alpha l^\rho + \beta k^\rho\right]^{(1/\rho)-1} \alpha l^{\rho-1}}{\left[\alpha_0 + \alpha l^\rho + \beta k^\rho\right]^{(1/\rho)-1} \beta k^{\rho-1}} = \frac{\alpha l^{\rho-1}}{\beta k^{\rho-1}} = (\frac{\alpha}{\beta})(\frac{k}{l})^{1-\rho}$$

It says, if labor is increased by one unit, capital has to be decreased by $(\frac{\alpha}{\beta})(\frac{k}{l})^{1-\rho}$ units if output is to remain constant.

Compare the $MRTS_{lk}$ for the Cobb-Douglas and the CES production function

$$MRTS_{lk}^{CD} = (\frac{\alpha}{\beta})(\frac{k}{l})$$

$$MRTS_{lk}^{CES} = (\frac{\alpha}{\beta})(\frac{k}{l})^{1-\rho}$$

and note that are the same if $\rho = 0$. It turns out that the Cobb-Douglas is a special case of the CES. This is not obvious upon initial examination because if one plugs $\rho = 0$ into the CES production function, the function is undefined. However, it is possible to show that

$$\lim_{\rho \to 0} \left[\alpha_0 + \alpha l^\rho + \beta k^\rho\right]^{1/\rho} = \alpha_0 l^\alpha k^\beta$$

That is, as $\rho \to 0$, the CES approaches the Cobb-Douglas.

The CES is more general than the Cobb-Douglas (the C.D. is a restrictive case of the CES), but also more difficult to estimate.

1.5 Consider the following policy question

The government wants to know what will happen to the demand for labor if the minimum wage increase. As a group will it make unskilled workers better off, or worse off. It hires you, Ms. Economic Consultant, for \$5000 per day to answer the question. You need to estimate, on average, how easily capital substitutes for labor in production; that is, you need to determine.

$$MRTS_{lk} \text{ for the U.S. economy}$$

For better or worse (you only get \$5000 per day), you assume some aggregate production function for the U.S. economy.

For thousands of firms you collect data on x, k and l, and use this data to estimate the values of the parameters in the your production function (e.g. α_0, α, β, ρ, etc.) to get your best estimate of the $MRTS_{lk}(k^0, l^0)$ for the U.S. economy at its current input levels.

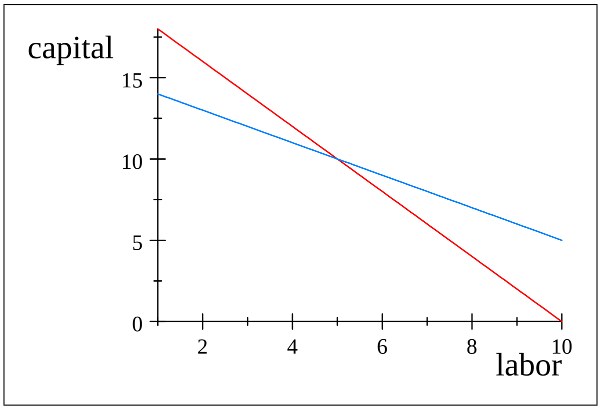
If this estimated $MRTS_{lk}(k^0, l^0)$ is small, an increase in the minimum wage will lead to a relatively large decrease in the demand for labor (put a lot of unskilled workers out of work). Why? Because it is relatively easy to substitute capital for labor: the less capital you need to replace an unskilled worker, the more likely you are to do it when the price of unskilled labor increases.

If this estimated $MRTS_{lk}$ is large, an increase in the minimum wage will lead to a relatively small decrease in the demand for labor (put some but not many unskilled workers out of work), Why? Because it is relatively difficult to substitute capital for labor.

Consider, for example, the following two simple example isoquants for some level of output (neither is CES or CD):

$$\begin{aligned} k &= 20 - 2l \\ k &= 15 - l \end{aligned}$$

They cross when $l = 5$ and $k = 10$. Obviously, these two isoquants represent different states of knowledge for producing a given level of x.



Blue is $k = 15 - l$, red is $k = 20 - 2$

If l decreases from 5 to 4 one only need to increase k by 1 unit if $k = 15 - l$:($MRTS_{lk} = 1$),but by 2 units if $k = 20 - 2$: ($MRTS_{lk} = 2$). The point of this example is simply to show that the $MRTS_{lk}$ is a measure of how easy it is to substitute away from labor.

Ceteris paribus, a dollar increase in the minimum wage will cause a decrease in the demand for unskilled workers (demand function is negatively sloped). The increase in unemployment increases as $MRTS_{lk}$ decreases.

1.6 Could one express $\left.\frac{dk}{dl}\right|_{dx=0}$, the slope of the isoquant in percentage terms.

That is

$$\left.\frac{\% \bigtriangleup k}{\% \bigtriangleup l}\right|_{dx=0}$$

$$\left.\frac{\% \bigtriangleup k}{\% \bigtriangleup l}\right|_{dx=0} = \lim_{\bigtriangleup l \to 0} \left.\frac{\frac{\bigtriangleup k}{k}}{\frac{\bigtriangleup l}{l}}\right|_{dx=0} = \frac{l}{k} \lim_{\bigtriangleup l \to 0} \left.\frac{\bigtriangleup k}{\bigtriangleup l}\right|_{dx=0} = \frac{l}{k} \left.\frac{dk}{dl}\right|_{dx=0}$$

1.7 Now let's generalize the concept an isoquant to the three-input case

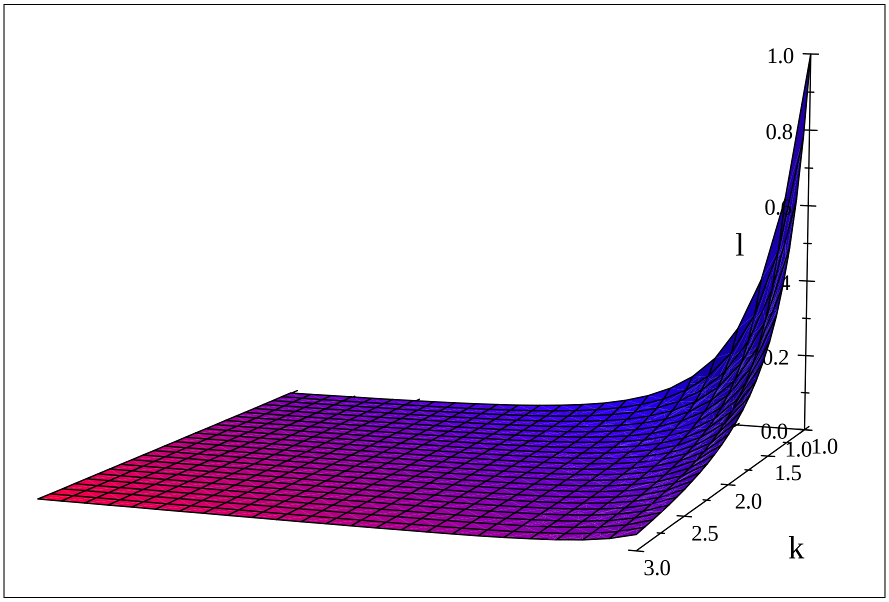
Let's produce gopher sausage. Assume

$$s = f(g, l, k) = A l^{\alpha} k^{\beta} g^{\gamma}$$

where s is the pounds of gopher sausage produced and g is the number of gophers used in the production process. Note that all three of the inputs are assumed *essential.*

Visualize the isoquant is three-dimensional labor-capital-gopher space. It might look something like a gigantic satelite dish.

For example it might look like



The isoquant for the output level s^0 consists of all those combinations of l, k, and g that are just capable of producing s^0 pounds of sausage. That is the isoquant is

$$L(s^0) \equiv (l, k, g) : f(g, l, k) = s^0,\ l, g, k \geq 0$$

Is the concept of a marginal rate of substitution still well defined? Yes - but between any pair of inputs, holding the other inputs constant.

To see this, let's look at the total differential of the production function

$$ds = f_g dg + f_l dl + f_k dk$$

Along the isoquant

$$0 = f_g dg + f_l dl + f_k dk$$

What if one want to know the rate at which gophers and labor substitute for one another in the production of gopher sausage, hold the level of capital constant. In this case

$$0 = f_g dg + f_l dl$$

Rearranging, one gets

$$\left.\frac{dg}{dl}\right|_{\substack{ds=0\\dk=0}} = -\frac{f_l}{f_g} = -MRTS_{\text{lg}}$$

What is the $MRTS_{\text{lg}}$ for the specified Cobb-Douglas? In this case

$$f_l = \alpha A l^{\alpha-1} k^{\beta} g^{\gamma} = \alpha s l^{-1}$$

and

$$f_g = \gamma A l^{\alpha} k^{\beta} g^{\gamma-1} = \gamma s g^{-1}$$

So

$$MRTS_{\text{lg}} = \frac{\alpha s l^{-1}}{\gamma s g^{-1}} = \frac{\alpha}{\gamma}\frac{g}{l}$$

1.8 Now let's switch gears and look at economic applications of differentials to consumer theory

Go back and review the notes *consumer theory in a nutshell* that are on the web page.

Initially assume a world of only two goods x_1 and x_2. Further assume that Wilber's preference can be represented with some utility function

$$u = u(x_1, x_2)$$

Now let define an indifference curve.

> **definition**: the indifference curve for utility level u^0 consists of all the combination of x_1 and x_2 that, if consumed, will just achieve the utility level u^0. Formally
>
> $$I\left(u^0\right) \equiv \left[(x_1, x_2) : u(x_1, x_2) = u^0\right]$$

Indifference curves are to the theory of the consumer as isoquants are to the theory of the firm. One can think of the utility function as a production function for utility.

Note that one could also define an indifference curve in terms of a bundle (x_1^0, x_2^0). It is all the bundles of of x_1 and x_2 such that that

$$I(x_1^0, x_2^0) \equiv \left[(x_1, x_2) : (x_1, x_2) \sim (x_1^0, x_2^0)\right]$$

Note that this definition of the indifference curve does not require that the utility function be identified, only that the individual can rank bundles of goods.

Further note that the indifference curve associated with $u^0$is the boundary of the set of all bundles that produce a $u \geq u^0$. That is, the upper level-set for u^0 is

$$I_w\left(u^0\right) \equiv \left[(x_1, x_2) : u(x_1, x_2) \geq u^0\right]$$

1.8.1 Consider the total differential of the utility function

$$du = f_{x_1} dx_1 + f_{x_2} dx_2$$

But, as one moves along an indifference curve, $du = 0$. So

$$0 = u_{x_1} dx_1 + u_{x_2} dx_2$$

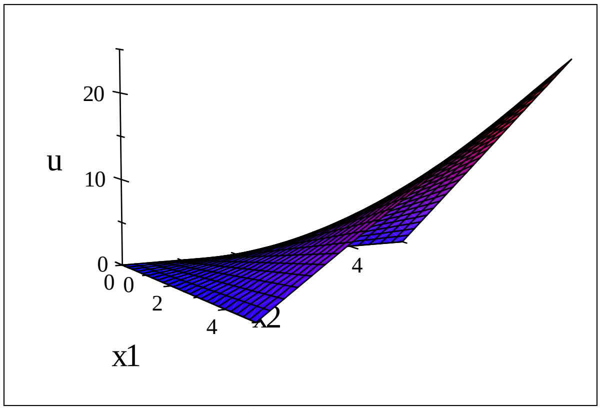
Rearranging,

$$\left. \frac{dx_2}{dx_1} \right|_{du=0} = -\frac{u_{x_1}}{u_{x_2}} \equiv -MRS_{x_1 x_2}$$

where $MRS_{x_1 x_2}$ is the marginal rate of substituion of x_1 for x_2. It tells us, loosely speaking, how much the consumption of x_2 must decreae to hold utility constant when x_1 increases by one unit. It is the rate at which the individual is willing to substitue one good for another. It is completely determined by the individual's preferences.

Assume Herkimer Snerd's utility function is

$$u = u(x_1, x_2) = x_1 x_2$$



$u = u(x_1, x_2) = x_1 x_2$

Herk is a simple guy.

We have already determined that

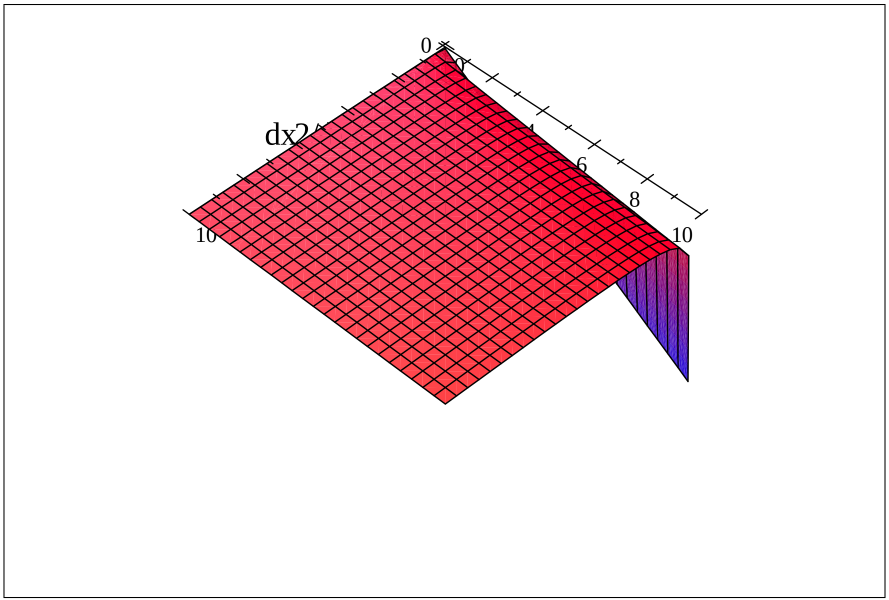
$$\left. \frac{dx_2}{dx_1} \right|_{du=0} = -\frac{f_{x_1}}{f_{x_2}} \equiv -MRS_{x_1 x_2}$$

So for Herk, we just need to calcuate and plug in f_{x_1} and f_{x_2}.

$$f_{x_1} = x_2$$
$$f_{x_2} = x_1$$

So, for Herk

$$\left.\frac{dx_2}{dx_1}\right|_{du=0} = -\frac{x_2}{x_1}$$



$\left.\frac{dx_2}{dx_1}\right|_{du=0} = -\frac{x_2}{x_1}$

Note that his $MRS_{x_1x_2}$ is dependent on the amount of the two goods he is currently consuming. For example, if $x_1^0 = 4$ and $x_2^0 = 2$, $MRS_{x_1x_2}(4,2) = \frac{1}{2}$. That is, he would substitue 1/2 units of x_2 for one unit of x_1.

However, if $x_1^0 = 2$ and $x_2^0 = 4$, $MRS_{x_1x_2}(2,4) = 2$. That is, he would substitue 2 units of x_2 for one unit of x_1.

1.8.2 Now consider another example. Herk stayed up all night watching infomercials, and his preferences are now

$$u = u(x_1, x_2) = a_1 x_1^\beta + a_2 x_2^\beta$$

In which case

$$\left.\frac{dx_2}{dx_1}\right|_{du=0} = -\frac{u_{x_1}}{u_{x_2}} = -\frac{\beta a_1 x_1^{\beta-1}}{\beta a_2 x_2^{\beta-1}} = -\frac{\alpha_1}{\alpha_2}\left(\frac{x_1}{x_2}\right)^{\beta-1}$$

Herk is now a much more complicated guy. But, what if $\beta = 1$? Note that $u = u(x_1, x_2) = a_1 x_1^\beta + a_2 x_2^\beta$ is a CES utility function.

1.8.3 Let's consider one more consumer application of total differentials.

Consider the consumer's budget constraint assuming two goods and exogenous income and prices

$$y \geq p_1 x_1 + p_2 x_2$$

Along the boundary of the budget set (the budget line)

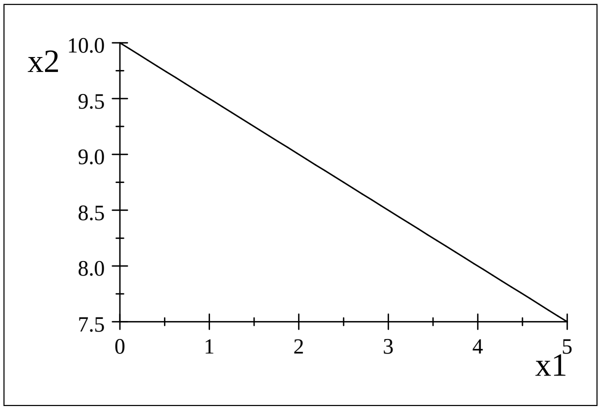
$$y = p_1 x_1 + p_2 x_2$$

If more is always preferred to less, the individuals will operate on the boundary of his budget set.

Solve for x_2 as a function of x_1

$$x_2 = \frac{y}{p_2} - \left(\frac{p_1}{p_2}\right) x_1$$

For example if $y = 10$, $p_2 = 10$ and $p_1 = 5$, $x_2 = 10 - .5x_1$ and the graph of the budget line is



One can derive the slope of the budget line by taking the partial derivative of the last function wrt x_1. This will tell us the rate the market allow the individual to substitute (trade) x_1 for x$_2$.

$$\frac{\partial x_2}{\partial x_1} = -\frac{p_1}{p_2}$$

It is the negative of the price ratio. This makes sense. For example, if $p_2 = 2$ and $p_1 = 1$, one unit x_1 trades for 1/2 units of x_2.

Note that

$$\frac{\partial x_2}{\partial x_1} = \left.\frac{dx_2}{dx_1}\right|_{dy=0}$$

so one could also derive the slope of the budget line using the concept of a total differential. Do it (remember that p_1 and p_2 are exogenous). Along the budget line $dy = 0$, so

$$0 = dy = p_1 dx_1 + p_2 dx_2$$

Rearranging, one gets

$$\left.\frac{dx_2}{dx_1}\right|_{dy=0} = -\frac{p_1}{p_2}$$

Compare $\left.\frac{dx_2}{dx_1}\right|_{dy=0}$ and $\left.\frac{dx_2}{dx_1}\right|_{du=0}$. The first is the rate at which the market allows the individual to substitue for good for another. This rate is exogenous to the individual. The second is the rate at which the individual is willing to substitute one good for another. This one is a function of the individual's preferences.