1 Economic applications of differentials and total differentials

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1.1 The basic theory

The differential dy is a "measure" of how much y changes

Differentials are very useful for deriving isoquants, indifference curves, and other boundaries of level sets.

Assume

$$y = f(x)$$

where x is, for now, a scalar. Note that for a change in x, denoted $\triangle x$,

$$\triangle y = \frac{\triangle y}{\triangle x} \triangle x$$

But as $\triangle x \rightarrow 0$, $\frac{\triangle y}{\triangle x} \rightarrow \frac{dy}{dx}$. Therefore, for "small" changes in x,

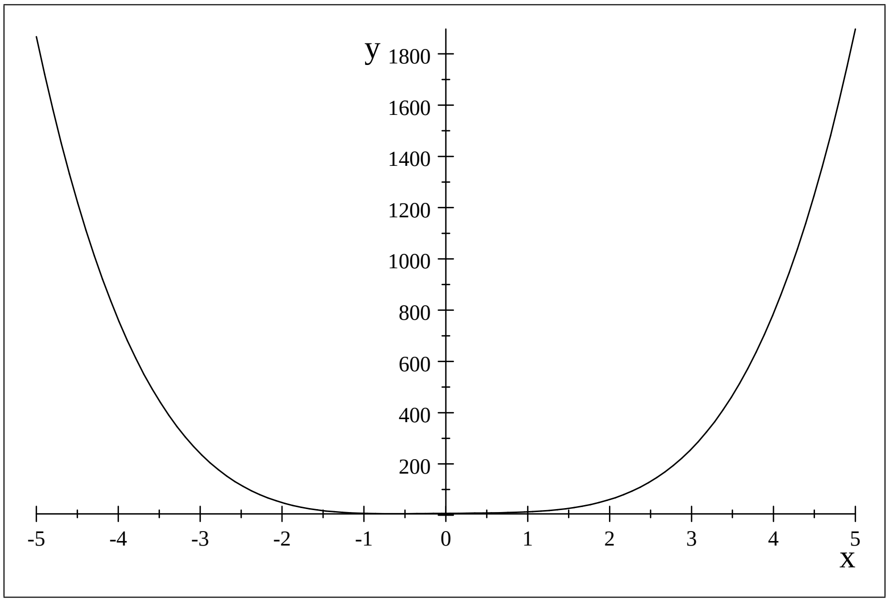
$$dy \approx \frac{dy}{dx} dx = f'(x) dx$$

and in general $\frac{dy}{dx} dx$ is an approximation to how much y changes, dy, when x changes, dx. Write this $dy \approx \frac{dy}{dx} dx = f'(x) dx$

This is the **differential** of y.

Consider a graphical example

Find dy when $y = f(x) = 3x + 3x^4 + 7$



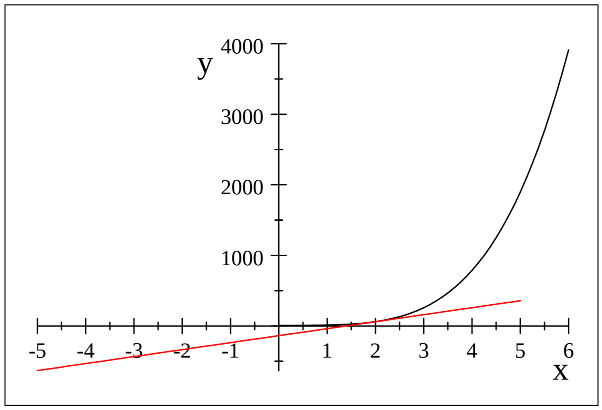
In this case

$$f'(x) = 3 + 12x^3$$

So

$$dy \approx f'(x)dx = (3 + 12x^3)dx$$

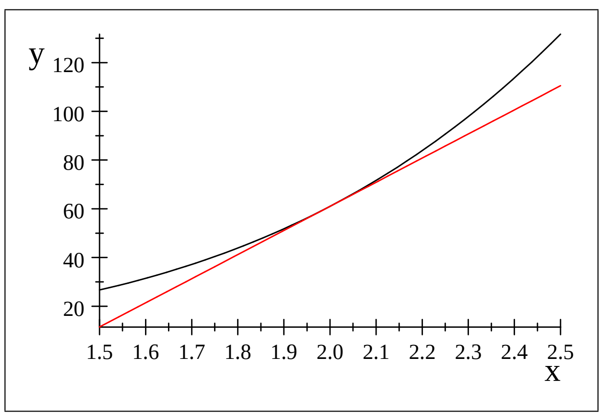
Using this differential, what is our best guess of dy when x increases when 2 to 4. The derivative of the function at $x = 2$ is $3 + 12(2)^3 = 99$ So a guess of how much y increases from x increases from 2 to 4 is 198 (99×2). How much does y really increase? At $x = 2$, y is $(2) + 3(2)^4 + 7 = 61$, and at $x = 4$, y is $3(4) + 3(4)^4 + 7 = 787$. The difference is 726. In this case, the approximation is inaccurate.



The tangent line at $x = 2$ is $y = -137 + 99x$, which has a slope of 99.

Note that this approximation increases (decreases) in accuracy as dx decreases (increases). For this example, figure it out for x increasing from $x = 2.0$ to $x = 2.1$.

One can see the approximation it much better for the change from 2.0 to 2.1 than it was for 2 to 4.



Note that one might interpret $\frac{dy}{dx}$ as the ratio of two differentials dy and dx.

1.2 Now consider the more general case where

$$y = f(x_1, x_2)$$

In which case, we refer to dy as the *total differential*

$$dy \approx \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 = f_{x_1}(x_1, x_2) dx_1 + f_{x_2}(x_1, x_2) dx_2$$

Note that if $dx_2 = 0$, the total differential simplifies to

$$dy|_{dx_2=0} \approx \frac{\partial y}{\partial x_1} dx_1$$

,which is a simple differential. Rearranging this, one gets

$$\left.\frac{dy}{dx_1}\right|_{dx_2=0} \approx \frac{\partial y}{\partial x_1}$$

That is, the partial derivative $\frac{\partial y}{\partial x_1}$ can be interpreted as the ratio two differentials dy and dx_1, holding x_2 constant.

Note that if $dx_1 = 0$, the total differential simplifies to $dy \approx \frac{\partial y}{\partial x_2} dx_2$, which is also a simple differential.

Let's calculate a total differential. Assume

$$z = 3x^2 + xy - 2y^3$$

$$dz \approx \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

where

$$\frac{\partial z}{\partial x} = 6x + y$$

and

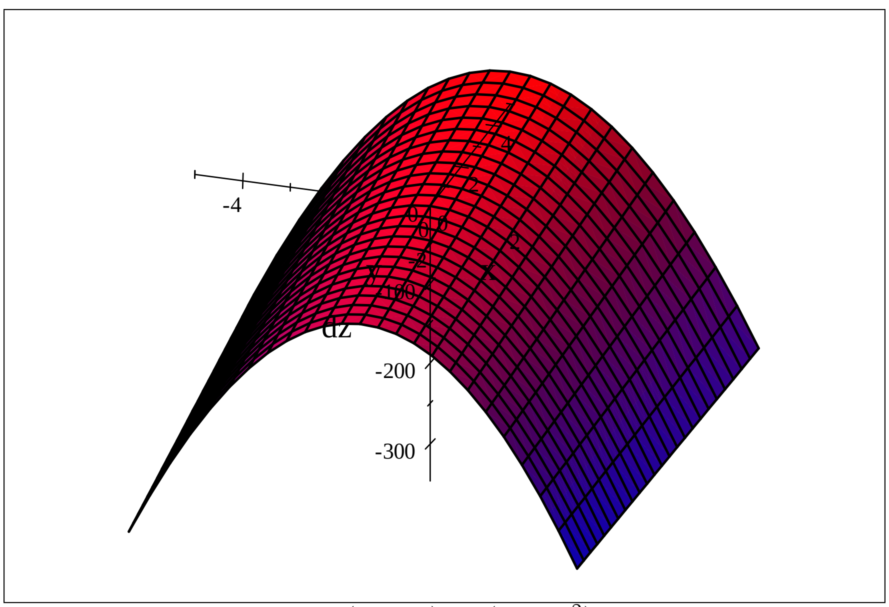
$$\frac{\partial z}{\partial y} = x - 6y^2$$

so

$$dz \approx (6x + y)dx + (x - 6y^2)dy$$

For example if $dx = 1$ and $dy = 2$ this simplifies to $dz \approx (6x+y)1+(x-6y^2)2$

Graphing this, how much z is estimated to change, using the differential, when x changes by 1 and y changes by 2 depends a lot on where you start. For example, if initially $x = y = 0$, $dz = 0$, but if $x = 4$ and $y = 2$, $(6(4) + 2)1 + (4 - 6(2)^2)2 = -14$



$dz \approx (6x + y)1 + (x - 6y^2)2$

Figure out the dz for a change from $(4,3)$ to $(5,5)$.

In this case

$$\begin{aligned} dz &\approx (6x+y)1+(x-6y^2)2 \\ &= ? \end{aligned}$$

You do the next one. Find dy assuming

$$y = f(x_1, x_2) = \frac{x_1}{x_1 + x_2} = x_1(x_1 + x_2)^{-1}$$

In this case

$$\begin{aligned} dy &\approx \left[(x_1 + x_2)^{-1} - x_1(x_1 + x_2)^{-2}\right] dx_1 + \left[-x_1(x_1 + x_2)^{-2}\right] dx_2 \\ &= \left[\frac{1}{(x_1 + x_2)} - \frac{x_1}{(x_1 + x_2)^2}\right] dx_1 - \frac{x_1}{(x_1 + x_2)^2} dx_2 \\ &= \left[\frac{(x_1 + x_2)}{(x_1 + x_2)^2} - \frac{x_1}{(x_1 + x_2)^2}\right] dx_1 - \frac{x_1}{(x_1 + x_2)^2} dx_2 \\ &= \frac{x_2}{(x_1 + x_2)^2} dx_1 - \frac{x_1}{(x_1 + x_2)^2} dx_2 \end{aligned}$$