

1 Econ 4808: Economic Applications of Constrained Optimization

May 1, 2012

Note that there are a lot of problems on this review sheet but that many of them are the same problem. For example, there are four or five questions in a row on determining the profit maximizing level of output for a monopolist. There is also a set of problems on profit maximization for the competitive firm and a bunch of consumer demand problems. You get the idea. Isolate each set (for example profit max problems for the monop vs profit max for the competitive firm) and note how they differ and how they are the same. Keep in mind that there is only a small number of types of problems that I can ask.

Review consumer theory and the theory of the firm in your intermediate micro text, and the class notes on these topics.

1. Describe in general terms how one would search for the global max or min of a **twice-differentiable** function of one variable, $f(x)$, $a \leq x \leq b$. What are the potential pitfalls of assuming the max (min) is at a level of x , x^o , where $f_x(x^o) = 0$. Explain

Answer: One needs to look everywhere for the global max. A necessary condition for global **interior** max is that $f(x^o) = 0$, so find all the values of x such that $f_x(x^o) = 0$ - these are the *critical* values of x .

Check second-order conditions and toss out all the candidates that are not local max; that is, toss all those critical values of x that do not have $f_{xx}(x^o) \leq 0$. The remaining x_o are all local interior max.

The largest one (there might only be one) is the global interior maximum. Now check to make sure that the global maximum is not one of the end points. That is, check the end points (a and b) to see if the value of $f(x)$ at one of the end points is greater than the value of the function at the global interior max. If one is, it is the global max; if not, the global interior max is the global max. (Note that if there is no limits on the range of x there are no corners, and the global interior max is the global max because all values of x are interior values.)

As an aside, and assuming $a \leq x \leq b$, the "the global interior max might not be the highest point in the interior of x . In which case, none of the point(s) that are higher are the global interior maximum. (Can you show me a graphical example of this case?)

What are the potential pitfalls of assuming the max (min) is at a level of x , x^o , where $f_x(x^o) = 0$. It might not be a local interior max. It could, for example, be a minimum. And, even if it is a local interior max, it might not be the global interior max. And, even if it is a global interior max, it might not be a global max.

Further note that the interior max might be at a point where the function is not twice differentiable. In the question we assumed twice differentiability.

2. Continue to assume that $f(x)$ is twice differentiable. Assume that you have found an x, x^o , where $f_x(x^o) = 0$. Identify an additional condition that is sufficient, but not necessary, for x^o to be a local interior max. Explain. Why isn't your condition necessary?

answer: $f_{xx}(x) < 0 \forall x$ is one possibility. It is not necessary because could be a lots of x^o that maximize a function but where $f_{xx}(x) \not< 0 \forall x$. Consider a case where $f_{xx}(x^o) < 0$ but $f_{xx}(x) \not< 0 \forall x$. Or consider a case, where at the local maximum, $f_x(x^o) = 0$ and $f_{xx}(x^o) = 0$. What is going on with this second case? Consider $f(x) = 5$.

3. Find the value(s) of x that maximize or minimize $u(x) = 9 - (x - a)^2 - 2(x - b)^2$. Now do the same for $u(x) = \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{2}{3}x + 1$.

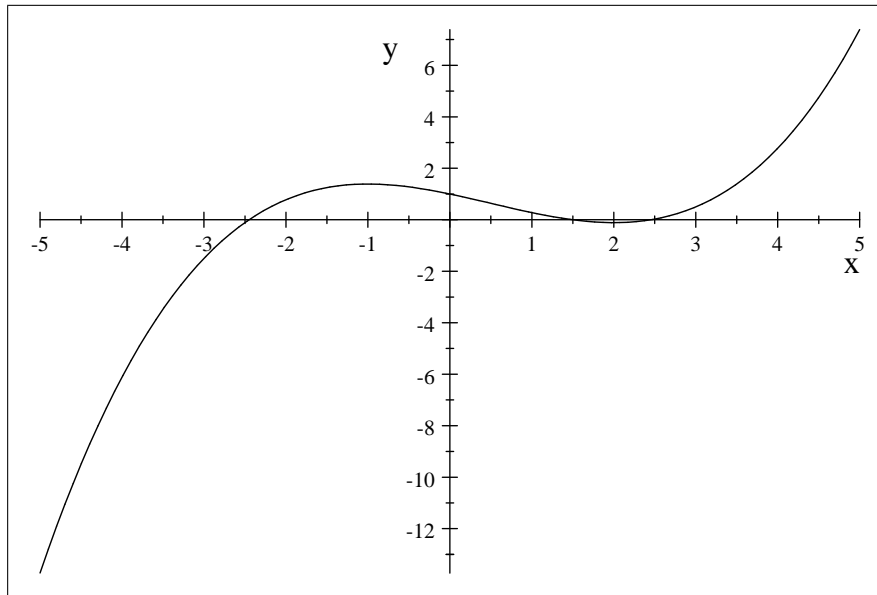
answer: Look for critical points. First find

$$\begin{aligned} u'(x) &= -2(x - a)1 - 4(x - b)1 \\ &= 2a + 4b - 6x \end{aligned}$$

. To find the critical points set this equal to zero and solve for x : $2a + 4b - 6x = 0$, Solution is: $x^o = \frac{1}{3}a + \frac{2}{3}b$. Is the function maximized or minimized at this point? $u''(x) = -6$, so the function is maximized at this point. Look for critical points. First find

$$u'(x) = \frac{1}{3}x^2 - \frac{1}{3}x - \frac{2}{3}$$

. To find the critical points set this equal to zero and solve for x : $\frac{1}{3}x^2 - \frac{1}{3}x - \frac{2}{3} = 0$, Solution is: $-1, 2$, so there are two critical points. Now look at $u'(x) = \frac{2}{3}x - \frac{1}{3}$, which can be either positive or negative depending on the value of x . Evaluate $u'(x) = \frac{2}{3}x - \frac{1}{3}$ at $x^o = -1$ and $x^o = 2$; $u'(-1) = \frac{2}{3}(-1) - \frac{1}{3} = -1.0$ and $u'(2) = \frac{2}{3}(2) - \frac{1}{3} = 1.0$. So, the function is maximized at $x = -1$ and minimized at $x = 2$. Consider a graph of the second function



4. Describe, in words, in words, a simple theory to explain the market behavior of an individual consumer. Now describe this theory in general functional notation. Describe both in words, and functional notation, the solution to the consumer's choice problem. Note that answering this question does not require that you do an actual algebra or calculus.
5. Describe in words and in mathematical notation the production manager's problem and its solution. Make sure to define all of your terms. Then relate the production manager's problem to the firm's cost function.

answer: The production manager's problem is to determine how much of each input to purchase to minimize the cost of producing a given level of output and exogenous costs. That is, her problem is to find the conditional input demand functions, $l_c^d = l_c^d(x, w, r)$ and $k_c^d = k_c^d(x, w, r)$ where x is units of output and w and r are the exogenous prices of labor and capital. She is also constrained by the state of technical knowledge for producing x . $l_c^d = l_c^d(x, w, r)$, for example, identifies the number of units the production manager wants to hire to minimize the cost of producing x units of output given w and r . Mathematically, the production manager's problem is

$$\begin{aligned} \min_{wrt\ l\ k} \quad & e = wl + rk \\ \text{subject to } x = & f(k, l) \end{aligned}$$

where $x = f(k, l)$ identifies the maximum amount of x that can be produced as a function of the amounts of labor and capital used. Relating all

of this to the cost function, since expenditures on the two inputs are, by definition $e = wl + rk$,

$$\begin{aligned} e^* &= wl_c^d + rk_c^d \\ &= wl_c^d(x, w, r) + rk_c^d(x, w, r) \\ &= c(x, w, r) \end{aligned}$$

That is, the cost function is another way to describe the solution to the production manager's problem. Note, as an aside that by Shepard's lemma

$$\frac{\partial c(x, w, r)}{\partial w} = l_c^d(x, w, r)$$

and

$$\frac{\partial c(x, w, r)}{\partial r} = k_c^d(x, w, r)$$

6. Assume Wilbur's utility function is $u = x_1x_2x_3$. Further assume that the law dictates that Wilbur consume two units of x_2 for every unit of x_1 . Determine Wilbur's demand function for good 1. In this part of the question do not worry about the second-order conditions for utility maximization. For now assume that the critical value of x_1 that you derived maximizes utility. Hint: Start by turning Wilbur's problem into an unconstrained problem in one variable. Explain, in words, all the steps in your derivation of his demand function for good 1. Now derive Wilbur's demand functions for goods 2 and 3. How many units of the three goods will he choose to purchase if his income is \$72, $p_1 = 1$, $p_2 = .5$ and $p_3 = 7$.
7. Discuss the role of marginal utility in demand theory.
8. Discuss the distinction between cardinal and ordinal preferences. As part of your answer, define both. Discuss the representation of preferences with a utility function.
9. Assume Wilbur's preferences can be represented by the utility function $u(x_1, x_2)$. Now assume some other utility function $U(x_1, x_2)$. List a set of necessary and sufficient conditions on $U(x_1, x_2)$ in terms $u(x_1, x_2)$ such that both functions represent the same preferences. As part of your answer, define preferences and discuss what it means for Wilbur to have preferences.
10. (5 points) Consider a world of two goods x_1 and x_2 where $x_1 > 1$ and $x_2 > 1$. Consider the the following two utility functions

$$u = x_1^\alpha x_2^\beta \text{ where } \alpha, \beta > 0$$

and

$$U = 14 + 5(\alpha \ln x_1 + \beta \ln x_2)^2$$

Convince me that these two utility functions will generate the same demand functions.

Answer: The easiest way to demonstrate this is to show that both utility functions represent the same preferences. The second one is just an increasing monotonic transformation of the first. Start with $u = x_1^\alpha x_2^\beta$. Note that $\ln u = (\alpha \ln x_1 + \beta \ln x_2)$. Logging both sides of $u = x_1^\alpha x_2^\beta$ does not change the ranking. Neither does multiplying by 5 or adding 14 to the result.

Note that

$$\frac{\partial U}{\partial u} = \frac{\partial[14 + 5(\alpha \ln x_1 + \beta \ln x_2)^2]}{\partial \ln u} \frac{\partial \ln u}{\partial u}$$

but

$$\ln u = \alpha \ln x_1 + \beta \ln x_2$$

So

$$\begin{aligned} \frac{\partial U}{\partial u} &= \frac{\partial[14 + 5(\alpha \ln x_1 + \beta \ln x_2)^2]}{\partial \ln u} \frac{\partial \ln u}{\partial u} \Rightarrow \\ \frac{\partial U}{\partial u} &= \frac{\partial[14 + 5(\ln u)^2]}{\partial \ln u} \frac{1}{u} = \frac{10 \ln u}{u} \end{aligned}$$

what is the point of showing this derivative?

11. Define, in words, the utility function $u(x_1, x_2)$, $x_1, x_2 > 0$.

Define, in words, the indifference curve associated with u^0 units of utility.

Using the concept of a total differential, determine $\left. \frac{dx_1}{dx_2} \right|_{du=0}$. Show all your work and explain, in words each step of your derivation.

Now assume

$$u = u(x_1, x_2) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2^2$$

What is the $MRS_{x_2 x_1}$ when $u = 5 + 4x_1 + 8x_2^5$?

What does this mean when $x_2^0 = 9$?

answer:

The utility function associates a number with each bundle of the two goods, such that if the individual is indifferent between two bundles the function assigns the same number to each, and if a bundle is preferred to another bundle, the function assigns a larger number to the preferred bundle.

The indifference curve associated with the utility level u^0 consists of all those bundles of the two goods that just achieve the utility level u^0

The total differential of the utility function is

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2$$

along an indifference curve, utility does not change, so

$$0 = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2$$

Solve this to get

$$\left. \frac{dx_1}{dx_2} \right|_{du=0} = - \frac{\frac{\partial u}{\partial x_2}}{\frac{\partial u}{\partial x_1}}$$

Given $u(x_1, x_2) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2^\gamma$

$$\left. \frac{dx_1}{dx_2} \right|_{du=0} = - \frac{\frac{\partial u}{\partial x_2}}{\frac{\partial u}{\partial x_1}} = - \frac{\gamma \alpha_2 x_2^{\gamma-1}}{\alpha_1}$$

When $u = 5 + 4x_1 + 8x_2^5$

$$\begin{aligned} \left. \frac{dx_1}{dx_2} \right|_{du=0} &= - \frac{\gamma \alpha_2 x_2^{\gamma-1}}{\alpha_1} \\ &= - \frac{.5(8)x_2^{-.5}}{4} \\ &= -x_2^{-.5} \end{aligned}$$

When x_2 is increases by one unit x_1 must decrease by $x_2^{-.5}$ units. $MRS_{x_2 x_1} = x_2^{-.5}$

Evaluate this when $x_2^o = 9$, $9^{-.5} = 0.33333$. This means that when the individual is consuming 9 units of x_2 , if x_2 is increased by 1 unit the consumption of x_1 must fall by $1/3$ units to keep utility constant.

12. Assume that an individual's preferences can be described by the following utility function

$$u = u(x_1, x_2) = ax_1 + bx_2^{1/2}$$

Assume more is always preferred to less. What does this assumption imply about a and b ? Explain how you would go about deriving the individual's demand functions for the two goods. Don't solve the problem rather explain, in steps, how you would solve the problem. Now try and derive the individual's demand function for good 2. If you get a solution, discuss whether or not you are sure that you have found the demand functions.

What happens to the demand for good 2 if the price of good 1 increases? Does it increase or decrease and by how much?

answer: More is preferred to less requires that the marginal utility of each good is always positive. That is,

$$\frac{\partial u}{\partial x_1} = a > 0$$

and

$$\frac{\partial u}{\partial x_2} = .5bx_2^{-.5} > 0$$

so a and b must both be positive.

$$\max_{\text{wrt } x_1, x_2} ax_1 + bx_2^{1/2}$$

subject to

$$y = p_1x_1 + p_2x_2$$

First turn the problem into an unconstrained maximization problem in one variable, x_2 by solving the budget constraint for x_1 and substituting the result into the objective function. One obtains

$$\max u = a \left(\frac{y - p_2x_2}{p_1} \right) + bx_2^{1/2}$$

First look for critical point(s)

$$\frac{\partial u}{\partial x_2} = -\frac{ap_2}{p_1} + .5bx_2^{-.5}$$

Set this equal to zero and solve for x_2 , solution is: $\left\{ x_2^o = 0.25b^2 \frac{p_1^2}{a^2p_2^2} \right\}$ This is possibly the demand function. To check to see if utility is maximized at this level of x_2 check to see if

$$\frac{\partial^2 u}{\partial x_2^2} = -.25bx_2^{-1.5}$$

is negative when $x_2^o = 0.25b^2 \frac{p_1^2}{a^2p_2^2}$. Since $-.25bx_2^{-1.5} < 0$ for all $x_2 > 0$, it is negative at x_2^o , so the demand function is

$$x_2^d = 0.25b^2 \frac{p_1^2}{a^2p_2^2}$$

To determine what happens to it when p_1 increases

$$\frac{\partial x_2^d}{\partial p_1} = \frac{\partial \left(0.25b^2 \frac{p_1^2}{a^2p_2^2} \right)}{\partial p_1} = 0.5b^2 \frac{p_1}{a^2p_2^2} > 0$$

That is, demand for x_2 increases by $0.5b^2 \frac{p_1}{a^2p_2^2}$ when p_1 marginally increases.

13. Assume that an individual's preferences can be described by the following utility function

$$u = u(x_1, x_2, x_3) = ax_1 + bx_2^{1/2} + cx_3^{1/2}$$

Assume more is always preferred to less. What does this assumption imply about a, b , and c ? Explain how you would go about deriving the individual's demand functions for the three goods. Don't solve the problem rather explain, in steps, how you would solve the problem. Now try and solve the problem. If you get a solution, discuss whether or not you are sure that you have found the demand functions.

What happens to the demand for good 1 if the price of good 3 increases?

Now answer the above question, but assume there is a law that says an individual has to consume the same number of units of goods 1 and 2.

14. Assume that an individual's preferences can be described by the following utility function

$$u = u(x_1, x_2, x_3) = x_1x_2 + x_3$$

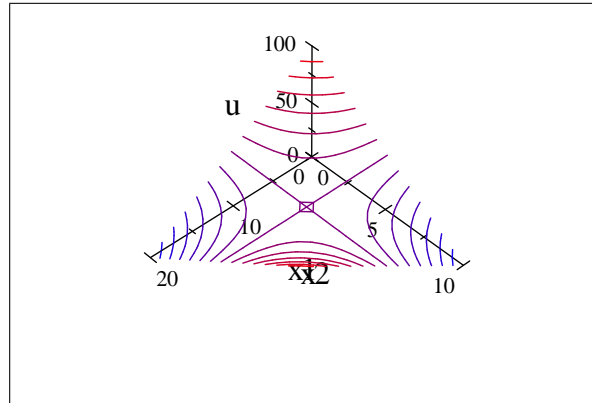
Derive the demand functions for the three goods. (Don't worry about checking the second order conditions for a max.)

answer: We want to maximize $u = x_1x_2 + x_3$ subject to $m = p_1x_1 + p_2x_2 + p_3x_3$. Note that it this utility function implies that the individual will spent all of his income, m . I will use substitution. Solve the budget constraint for x_3 , $m = p_1x_1 + p_2x_2 + p_3x_3$, Solution is: $\left\{x_3 = \frac{m - p_1x_1 - p_2x_2}{p_3}\right\}$. I chose x_3 because it is the additive term. Substitute for x_3 in the utility function to get $u = x_1x_2 + \frac{m - p_1x_1 - p_2x_2}{p_3}$. Look for stationary points. $\frac{d}{dx_1} \left[x_1x_2 + \frac{m - p_1x_1 - p_2x_2}{p_3} \right] = \left[-\frac{-x_2p_3 + p_1}{p_3} \right]$, $\frac{d}{dx_2} \left[x_1x_2 + \frac{m - p_1x_1 - p_2x_2}{p_3} \right] = \frac{x_1p_3 - p_2}{p_3}$. Set each of these partials to zero and solve. With this simple utility function each can be solved separately: $\left[-\frac{-x_2p_3 + p_1}{p_3} \right] = 0$, Solution is: $\left\{x_2^* = \frac{p_1}{p_3}\right\}$ and $\frac{x_1p_3 - p_2}{p_3} = 0$, Solution is: $\left\{x_1^* = \frac{p_2}{p_3}\right\}$. And by substituting for x_1 and x_2 in the budget constraint one obtains $x_3^* = \frac{m}{p_3} - \frac{2p_1p_2}{p_3^2}$.

If one wanted to check the second order conditions, one would want to show that

$$\begin{aligned} & u_{x_1x_1}(x_1, x_2)v_1^2 + u_{x_2x_2}(x_1, x_2)v_2^2 + 2u_{x_1x_2}(x_1, x_2)v_1v_2 \\ &= 0 + 0 + 2(1)v_1v_2 \end{aligned}$$

is non-positive. One can't. It is positive as long as v_1 and v_2 are both positive or both negative. So moving away from the stationary point in either the NE direction (increasing x_1 and x_2 , decreasing x_3) or the SW direction will increase utility. It looks like the stationary point that we have found is not a max, so we have not found the demand functions. Let's graph the utility function to see if we can figure out what is going on. To do this need to assume values for income and prices. For example $x_1x_2 + \frac{100 - 5x_1 - 10x_2}{1}$



It look, to me, like with these prices utility is maximized when everything is spent on x_3 , a corner solution.

15. Find the consumer's demand functions for x_1 and x_2 assuming $u = x_1x_2$ subject to

$$x_2 = \frac{y}{p_2} - \frac{p_1}{p_2}x_1$$

answer: the constraint into the objective function, one obtains

$$\max_{wrt x_1} g(x_1) = x_1 \left[\frac{y}{p_2} - \frac{p_1}{p_2}x_1 \right] = \frac{y}{p_2}x_1 - \frac{p_1}{p_2}x_1^2$$

Note that Note that we have turn an constrained optimization in two variables into an unconstrained optimization problem in one variable. The solution is the consumer's demand function for good 1,

$$x_1 = x_1(y, p_1, p_2)$$

How do we find the value of x , x^* , that maximizes utility subject to the constraints? We are looking for the top of the utility mountain. How can we find it? Maybe at the top of the mountain, $g'(x_1) = 0$. We call the value of x_1 , where $g'(x_1) = 0$, x_1^0 , that is, $g'(x_1^0) = 0$. It is often call a "critical" value. Find x_1^0

$$g'(x_1) = \frac{y}{p_2} - \frac{2p_1}{p_2}x_1$$

Setting this equal to zero and solving for x_1 , one obtains

$$\begin{aligned} 0 &= \frac{y}{p_2} - \frac{2p_1}{p_2}x_1 \\ \Rightarrow \frac{2p_1}{p_2}x_1 &= \frac{y}{p_2} \\ \Rightarrow x_1^0 &= x_1(y, p_1, p_2) = \frac{yp_2}{p_2p_12} = \frac{y}{2p_1} \end{aligned}$$

So, $x_1^0 = x_1(y, p_1, p_2) = \frac{y}{2p_1}$ is possibly the demand function for good one. If it is, it says that the individual should spend 1/2 of his or her income on each good. How would you make sure that x_1^0 is the quantity of x_1 that maximizes rather than minimizes utility? Maybe look at the second derivative of $g(x_1)$, $g''(x_1)$ at x_1^0 .

$$g'(x_1) = -\frac{2p_1}{p_2}$$

We want to evaluate this at x_1^0 . Note that it is the same constant, $-\frac{2p_1}{p_2}$, at every value of x , so $g'(x_1^0) = -\frac{2p_1}{p_2} < 0 \forall p_i > 0$. What does this tell us? that utility is maximized at

$$x_1^0 = x_1(y, p_1, p_2) = \frac{y}{2p_1}$$

Why? The individual's demand functions (Marshallian demand functions) are

$$x_1^0 = x_1(y, p_1, p_2) = \frac{y}{2p_1}$$

$$x_2^0 = x_2(y, p_1, p_2) = \frac{y}{2p_2}$$

The individual will spend 1/2 of his income on each good. This should not surprise us given the symmetrical way the two goods enter the demand function.

16. Find the individual's demand functions assuming $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ where $\alpha > 0$. Simplified by making $\alpha = .5$, so the problem is

$$\max_{wrt x_i, x_2} x_1^5 x_2^5 = (x_1 x_2)^5 \quad (1)$$

subject to

$$y = p_1 x_1 + p_2 x_2$$

Think about this utility function intuitively and compare it to $u = x_1 x_2$. The two commodities are treated symmetrically. Also note that

$x_i = \frac{.5y}{p_i}$ when $u = x_1 x_2$, and note that $(x_1 x_2)^5$ is a monotonic transformation of $x_1 x_2$. You should know what the answer is before you even begin to do any math. Draw some indifference curves, assign number and then take the square root of those numbers.

Now let's do the math. What are the steps? First step: turn the problem into an unconstrained problem in one variable. Do this by solving the budget constraint for x_2 and substituting the result into the object function. That is

$$x_2 = \frac{y - p_1 x_1}{p_2}$$

So, the objective function in terms of x_1 is

$$m = x_1^5 \left(\frac{y - p_1 x_1}{p_2} \right)^5$$

We want to maximize this without respect to x_1 . Start by looking for a critical value(s) for x_1 ; that is, value of x_1 where $\frac{\partial m}{\partial x_1}$

$$\frac{\partial \left[x_1^5 \left(\frac{y - p_1 x_1}{p_2} \right)^5 \right]}{\partial x_1} = \left[-0.5 \frac{-1.0y + 2.0p_1 x_1}{\sqrt{x_1} \sqrt{\left(-1.0 \frac{-1.0y + p_1 x_1}{p_2} \right) p_2}} \right] \text{ Now set this equal to}$$

zero and solve for x_1

$$\left[-0.5 \frac{-1.0y + 2.0p_1 x_1}{\sqrt{x_1} \sqrt{\left(-1.0 \frac{-1.0y + p_1 x_1}{p_2} \right) p_2}} \right] = 0,$$

Solution is $\left\{ x_1 = 0.5 \frac{y}{p_1} \right\}$. This is very interesting, it is the same critical value for x_1 that we got when we assumed $u = x_1 x_2$. Does that make sense. They represent the same preference ordering. Reread consumer theory in a nutshell. That means that we could have saved a lot of work by first seeing if we could simplify the functional form of the utility function without changing the preference ordering. In this case, we don't need to check to make sure utility is maximized at $x_1 = 0.5 \frac{y}{p_1}$. We already know that it is.

17. Assume that an individual's preferences can be described by the following utility function

$$u = u(x_1, x_2) = ax_1 + bx_2$$

Assume more is always preferred to less. Derive the individual's demand functions. Explain, in words, what you are doing and the logic of what you are doing. Think before you leap. Use a graph or graphs to make your argument.

answer: More is preferred to less requires that a and b are both positive. The individual's problem is to

$$\max_{\text{wrt } x_1, x_2} ax_1 + bx_2$$

subject to

$$y = p_1 x_1 + p_2 x_2$$

First turn the problem into an unconstrained maximization problem in one variable, x_2 by solving the budget constraint for x_1 and substituting the result into the objective function. One obtains

$$\max u = a \left(\frac{y - p_2 x_2}{p_1} \right) + bx_2$$

First look for critical point(s)

$$\frac{\partial u}{\partial x_2} = -\frac{ap_2}{p_1} + b$$

Stop and look at $\frac{\partial u}{\partial x_2}$ and note that in this case it is not a function of x_2 and depending on the values of a , b , p_2 and p_1 it will always be positive or negative. What does this mean? It means that maximum utility is either always increasing in x_2 or always decreasing. If $-\frac{ap_2}{p_1} + b < 0$ the individual should consume no x_2 and spend all of their income on x_1 . That is, if $-\frac{ap_2}{p_1} + b < 0$, $x_1^* = \frac{y}{p_1}$ and $x_2^* = 0$. If $-\frac{ap_2}{p_1} + b > 0$ the individual should consume no x_1 and spend all of their income on x_2 . That is, if $-\frac{ap_2}{p_1} + b > 0$, $x_2^* = \frac{y}{p_2}$ and $x_1^* = 0$. Mathematically speaking, the individual is at a corner solution, spending all of their income on one of the two goods, the good that gives the most bang for the buck. Think of it this way; the individual gets a utils from each unit of x_1 and if it costs p_1 per unit, the individual gets a/p_1 utils for each dollar spent on x_1 . Likewise he always gets b/p_2 utils for each dollar spent on x_2 . He will spend all of his money on the one that is larger. Note that this guy's indifference curves are straight lines; $\frac{dx_2}{dx_1} |_{du=0} = -\frac{a}{b}$, the two goods are perfect substitutes. Draw a picture of the indifference curves and the budget line.

18. Describe the theory of the competitive firm in terms of input quantities and what the theory predicts. Do this both in words and in functional notation. Note that answering this question does not require that you do any actual algebra or calculus.
19. Describe the theory of the competitive firm in terms of output and what the theory predicts. Do this both in words and in functional notation. Note that answering this question does not require that you do any actual algebra or calculus.
20. Assume a competitive firm produces product x using k and l . Further assume that x is sold at the parametric price p and that r and w are the parametric prices of k and l .

Define in words, the firm's longrun demand function for capital services, $k^d = k(p, w, r)$.

Derive the firm's longrun demand function for labor, l^d , and capital services, k^d , assuming

$$x = f(k, l) = k^{.5}l^{.5}$$

Don't worry about checking the second-order conditions.

answer: First write down profits as a function of k and l ,

$$\pi(k, l) = p(k^{.5}l^{.5}) - wl - rk$$

Look for critical the critical point(s). First take the partial derivatives $D_l(p(k^{.5}l^{.5}) - wl - rk) = 0.5\frac{k^{0.5}}{l^{0.5}}p - w$ and $D_k(p(k^{.5}l^{.5}) - wl - rk) = \frac{0.5}{k^{0.5}}l^{0.5}p - r$. The production function is symmetrical so it does not matter whether we start with k or l . Set the first partial to zero and solve for l , $0.5\frac{k^{0.5}}{l^{0.5}}p - w = 0$, Solution is: $l = \frac{1}{4}k\frac{p^2}{w^2}$ Plug this into $\frac{0.5}{k^{0.5}}l^{0.5}p - r = 0$ to get $\frac{0.5}{k^{0.5}}(\frac{1}{4}k\frac{p^2}{w^2})^{0.5}p - r = \frac{0.5}{k^{0.5}}(\frac{1}{4})^{0.5}k^{0.5}\frac{p}{w}p - r = 0.5(\frac{1}{4})^{0.5}\frac{p^2}{w} - r = 0.25p^2w^{-1} - r = 0$. Bells go off in your head at this point, $0.25p^2w^{-1} - r = 0$ is not a function of l , so there is not critical value of l , which tells us that the amount of labor the firm want to maximize its profits is either zero or infinity. But, we should have known this before we started the math: the production function is constant returns to scale, so the competitive firm will will want produce either zero or infinity, depending on whether the constant marginal cost is greater than or less than p .

Now let's do a problem with there is an interior solution. Assume now decreasing returns to scale. For example, $x = f(k, l) = k^{.5}l^{.2}$. First write down profits as a function of k and l ,

$$\pi(k, l) = p(k^{.5}l^{.2}) - wl - rk$$

Look for critical the critical point(s). First take the partial derivatives $D_l(p(k^{.5}l^{.2}) - wl - rk) = 0.2\frac{k^{0.5}}{l^{0.8}}p - w$ and $D_k(p(k^{.5}l^{.2}) - wl - rk) = \frac{0.5}{k^{0.5}}l^{0.2}p - r$. Set the first partial to zero and solve for $k^{0.5}$ (this because it will plug directly into the other first-order condition. Simplifying, $0.2\frac{k^{0.5}}{l^{0.8}}p - w = 0$ implying $0.2\frac{k^{0.5}}{l^{0.8}}p = w$ implying $k^{0.5} = 5wp^{-1}l^{0.5}$. Plug this into $\frac{0.5}{k^{0.5}}l^{0.2}p - r = 0$ to get $\frac{0.5}{5wp^{-1}l^{0.5}}l^{0.2}p - r = 0 = \frac{0.1}{l^{0.3}}\frac{p^2}{w} - r = 0$ which implies that $\frac{0.1}{l^{0.3}}\frac{p^2}{w} = r$ which implies that $\frac{1}{l^{0.3}} = \frac{10rw}{p^2}$ which implies $l = (l^{0.3})^{\frac{1}{0.3}} = (\frac{p^2}{10rw})^{\frac{1}{0.3}} = (\frac{p^2}{10rw})^{3.333}$. So, $l_{crit} = (\frac{p^2}{10rw})^{3.333}$. Let's hope this is the demand function for labor. Now find the critical value for capital. Plug l_{crit} into either of the first-order conditions to get k_{crit} . For example, $k_{crit}^{0.5} = 5wp^{-1}l_{crit}^{0.5} = 5wp^{-1}((\frac{p^2}{10rw})^{3.333})^{0.5}$. So, $k_{crit} = (5wp^{-1})^2((\frac{p^2}{10rw})^{3.333}) = \frac{25w^2}{p^2}((\frac{p^2}{10rw})^{3.333}) = 25w^2p^{-2}p^{6.666}10^{-1}r^{-3.333}w^{-3.333} = \frac{2.5p^{4.666}}{r^{3.333}w^{1.333}}$. Wow! So, let's hope this is the demand function for capital.

So, we better check second-order conditions to make sure. Look at $D_{ll}(p(k^{.5}l^{.2}) - wl - rk) = -0.16\frac{k^{0.5}}{l^{1.8}}p$ and $D_{kk}(p(k^{.5}l^{.2}) - wl - rk) = -\frac{0.25}{k^{1.5}}l^{0.2}p$ which are both negative so we most likely have a max. So, the demand functions for labor and capital are

$$l^d = l^d(p, w, r) = (\frac{p^2}{10rw})^{3.333}$$

and

$$k^d = k^d(p, w, r) = \frac{2.5p^{4.666}}{r^{3.333}w^{1.333}}$$

21. Assume a competitive firm sells its output x at the parametric price p and that it can purchase labor and capital at the parametric prices w and r . Further assume that the firm's cost function is

$$c = c(x, w, r) = x^2wr$$

Determine the profit maximizing level of output, x^s . Show all of your work and explain, in words, all of your steps. Derive a quantitative comparative static prediction about what happens to x^s if r increases.

22. Assume a competitive firm sells its output x at the parametric price p and that it can purchase labor and capital at the parametric prices w and r . Further assume that the firm's cost function is

$$c = c(x, w, r) = 3x^2w^{.7}r^{.2}$$

Determine the profit maximizing level of output, x^s . Show all of your work and explain, in words, all of your steps. Derive a quantitative comparative static prediction about what happens to x^s if r increases.

$$c = c(x, w, r) = 3x^2w^{.7}r^{.2}$$

so,

$$\pi(x) = px - 3x^2w^{.7}r^{.2}$$

First look for critical points, $D_x(px - 3x^2w^{.7}r^{.2}) = p - 6r^{.2}w^{0.7}x$. Set this equal to zero and solve for x to get $p - 6r^{.2}w^{0.7}x = 0$, Solution is: $x_0 = \left\{ \frac{1}{6} \frac{p}{r^{.2}w^{0.7}} \right\}$. Check to see if this critical point is a max. $D_{xx}(px - 3x^2w^{.7}r^{.2}) = -6r^{.2}w^{0.7} < 0$ for all values of x , so is negative when $x_0 = \left\{ \frac{1}{6} \frac{p}{r^{.2}w^{0.7}} \right\}$. Therefore the firms supply function is $x^s(p, w, r) = \left\{ \frac{1}{6} \frac{p}{r^{.2}w^{0.7}} \right\}$ What happens to this if r marginally increases? $D_r\left(\frac{1}{6} \frac{p}{r^{.2}w^{0.7}}\right) = -3.3333 \times 10^{-2} \frac{p}{r^{1.2}w^{0.7}} = -.0333 \frac{p}{r^{1.2}w^{0.7}}$. That is, if r marginally increases, the profit maximizing level of output will decline by $.0333 \frac{p}{r^{1.2}w^{0.7}}$

23. Assume a competitive firm sells its output x at the parametric price p and that it can purchase labor and capital at the parametric prices w and r . Further assume that the firm's cost function is

$$c = c(x, w, r) = x^{.5}wr$$

Determine the profit maximizing level of output, x^s . Show all of your work and explain, in words, all of your steps. Derive a quantitative comparative static prediction about what happens to x^s if r increases.

answer: Note that this cost function exhibits increasing returns to scale that is, the marginal cost of production $\left(\frac{d}{dx}(x^{.5}wr) = mc(x, w, r) = \frac{0.5}{\sqrt{x}}wr\right)$ decreases $\left(\frac{d^2}{dx^2}(x^{.5}wr) = \frac{dmc(x, w, r)}{dc} - \frac{0.25}{x^{\frac{3}{2}}}wr < 0\right)$ as output increases. The competitive firm, which can sell as much as it wants at the parametric price p , will want to produce infinite output. If you look for an interior point, you will find a critical point, but profits will be minimized at that point.

24. Assume a competitive firm sells its output x at the parametric price p and that it can purchase labor and capital at the parametric prices w and r . Further assume that the firm's cost function is

$$c = c(x, w, r) = xwr$$

Determine the profit maximizing level of output, x^s . Show all of your work and explain, in words, all of your steps.

25. Assume $\pi = px - c(x)$, where π is profits. Assume $p = 40$, and $c(x) = x^3 - 12x^2 + 60x$. Find the profit maximizing level of output. Don't forget to check the second-order conditions.

answer: Profits as a function of x are

$$\pi(x) = 40x - x^3 + 12x^2 - 60x$$

Look for a stationary point: $\frac{d}{dx}(40x - x^3 + 12x^2 - 60x) = -20 - 3x^2 + 24x = 0$, Solution is: $\{x = 4 + \frac{2}{3}\sqrt{21}\}$, $\{x = 4 - \frac{2}{3}\sqrt{21}\}$. $4 + \frac{2}{3}\sqrt{21} = 7.0551$, $4 - \frac{2}{3}\sqrt{21} = 0.94495$. There are two stationary points. Check the sign if $\frac{d^2}{dx^2}(40x - x^3 + 12x^2 - 60x) = -6x + 24$. This is positive if $x < 4$ and negative if $x > 4$. So profits are maximized when $x = 7.0551$ and minimized when $x = .94495$.

26. Assume the Snerd Corporation produces product x using k and l , where

$$x = f(k, l) = k^{.5}l^{.5}$$

Further assume that the firm purchases labor and capital at the parametric prices w and r . Derive the firm's conditional demand function for labor; that is, derive $l_c^d = l_c(x, w, r)$. Then derive the firm's conditional demand function for capital, $k_c^d = k_c(x, w, r)$. Note that the conditional demand function for labor identifies the amount of labor the firm will purchase to minimize the total cost of producing x , given w and r . You do not have to check the second order conditions for a max. What happens to the conditional demand for capital if the price of labor increases. Explain all of your steps in words. Now tell the reader what the firms cost function is and why?

answer: This is the production manager's problem. Min (wrt l) $wl + rk$ subject to $x = k^{.5}l^{.5}$. Note that there constant returns to scale in production. I will solve using substitution. $x = k^{.5}l^{.5} \Rightarrow x^2 = kl$, Solution is: $k = \frac{x^2}{l}$. Substitute this into the objective function to obtain $wl + r\frac{x^2}{l}$. Minimize this wrt l . Look for critical points. $\frac{d}{dl}(wl + r\frac{x^2}{l}) = -\frac{wl^2 + rx^2}{l^2} = 0$, Solution is: $\left\{l = \frac{1}{w}\sqrt{(wr)x}\right\}$, $\left\{l = -\frac{1}{w}\sqrt{(wr)x}\right\}$. Discard the negative one and check the second order conditions. $\frac{d^2}{dl^2}(wl + r\frac{x^2}{l}) = 2r\frac{x^2}{l^3}$, which is positive if r and l are positive, which they are by assumption. So,

$l^c(x, w, r) = \frac{1}{w}\sqrt{(wr)x} = x\left(\frac{r}{w}\right)^{.5}$ is the conditional demand function for labor. By symmetry $k^c(x, w, r) = x\left(\frac{w}{r}\right)^{.5}$ is the conditional demand function for capital. Note that both of these are linear in output. This follows from the fact that we assumed a production function that exhibited constant returns to scale. To determine what happens to the conditional demand for capital when the wage rate increases take the partial of $k = x\left(\frac{w}{r}\right)^{.5}$ with respect to w , that is, $\frac{d}{dw}\left(x\left(\frac{w}{r}\right)^{.5}\right) = 0.5\frac{x}{\sqrt{\left(\frac{w}{r}\right)r}} = .5xw^{-.5}r^{-.5} = \frac{.5x}{(wr)^{.5}}$

(different ways to express the partial derivative. So, if w marginally increases, holding x and w constant, costs will increase by $\frac{.5x}{(wr)^{.5}}$. Extra credit: Determining the cost function. Expenditures on inputs are, by definition the amount spent on labor and the amount spent on capital; that is, $e = wl + rk$. Cost minimizing expenditures to produce x given w and r are therefore

$$\begin{aligned} e^* &= wl^c + rk^c \\ &= wx\left(\frac{r}{w}\right)^{.5} + rx\left(\frac{w}{r}\right)^{.5} \\ &= x(wr^{.5}w^{-.5} + rw^{.5}r^{-.5}) \\ &= x(w^{.5}r^{.5} + w^{.5}r^{.5}) \\ &= 2xw^{.5}r^{.5} \\ &= c(x, w, r) \end{aligned}$$

27. (This one is long and has a lot of algebra.) Assume the Snerd Corporation produces product x using l and k , such that

$$x = k^\alpha l^{1-\alpha} \quad \text{where } 0 < \alpha < 1$$

Assume that the firm buys labor and capital at the parametric prices w and r . Derive the firms conditional demand functions for l and k ; that is, derive $l_c^d = l_c(x, w, r)$ and $k_c^d = k_c(x, w, r)$

28. Assume a competitive firm produces output x using only two inputs, l and k . Further assume you know the firm's cost function and that you have already derived its supply function. Describe how you might determine how much labor and capital the firm will want to hire.
29. Assume a competitive firm produces output x using only two inputs, l and k . Further assume you know the firm's production function and that you have already derived its demand functions for labor and capital. Describe how you might determine how much output the firm will want to produce.
30. Describe the theory of a monopoly firm in terms of output and what the theory predicts. Do this both in words and in functional notation. Note that answering this question does not require that you do any actual algebra or calculus.

31. Assume that the Gomer Corporation is the sole producer of gomers (i.e., it's a monopoly).

Assume that the aggregate demand function for gomers is

$$g = g(p) = 100 - .5p$$

where p is the price of gomers and g is the quantity demanded of gomers. Assume the cost function for producing gomers is

$$c(g) = g^3 - 12g^2 + 60$$

Determine the profit maximizing quantity of gomers. Explain each of your steps, show all of your work, and make sure to check the second-order conditions for profit maximization. What happens to the profit maximizing level of output if p increases. Is this a quantitative comparative static prediction? Can you sign it?

answer: Want to maximize profits wrt g . First need to define profits as a function of g . Begin by solving the demand function for p to get the inverse demand function $g = 100 - .5p$, Solution is: $\{p = -2.0g + 200.0\}$. Therefore, total revenue, as a function of g is $(-2.0g+200.0)g = -2.0(g - 100.0)g$. Therefore

$$\pi(g) = -2.0(g - 100.0)g - g^3 + 12g^2 - 60$$

Look for critical points $\frac{d}{dg}(-2.0(g - 100.0)g - g^3 + 12g^2 - 60) = 20.0g + 200.0 - 3.0g^2 = 0$, Solution is: $\{g = -5.4858\}, \{g = 12.153\}$. Discard the negative one. Check the second-order conditions $\frac{d^2}{dg^2}(-2.0(g - 100.0)g - g^3 + 12g^2 - 60) = 20.0 - 6.0g$, which is a negative number when $g = 12.153$. So the profit maximizing level of gomers is 12.153. Note that we could determine the profit maximizing price for this monopolist by plugging the profit maximizing output level into the firm's inverse demand function $p = -2.0g + 200.0$ and evaluating it at $g = 12.153$ to obtain $-2.0(12.153) + 200.0 = 175.69 = p_m$. Recollect that the monopolist gets to choose g and p , but not independently; that is, p and g are both endogenous variables in this model.

32. Assume a profit maximizing **monopolist**. The demand function for its output is

$$x(p) = b - ap, \text{ where } a, b, > 0$$

The cost function is

$$c(x, w, r) = x^2g(w, r)$$

where $g(w, r) > 0 \forall w, r > 0$, $g_w(w, r) > 0$ and $g_r(w, r) > 0$. Derive the profit maximizing level of output, x^s , as a function of w and r . That is derive $x^s = x^s(w, r)$. Don't forget to check the second order conditions. Why doesn't the profit maximizing level of output depend on p ? Does this model predict that the profit maximizing level of x will decrease if r increases? Yes or No and prove it.

Answer: The goal is to derive the profit maximizing level of output as a function of the exogenous input prices, w and r . Note that for the monopolist, price is endogenous. What is exogenous is the aggregate demand function for the firm's product. The first step is express the firm's profits as a function of x . Costs are already expressed in unit of x . Total revenue also needs to be expressed in units of x . Total revenue is price times quantity, but price is a function of quantity. That is, one want to express total revenue as a function of x .

$$tr(x) = p(x)x$$

$p(x)$ is the inverse demand function, which is obtained by solving $x = b - ap$ for p to obtain

$$x(p) = \frac{b - x}{a}$$

So,

$$tr(x) = p(x)x = \frac{(b - x)x}{a} = \frac{bx - x^2}{a}$$

So

$$\pi(x) = \frac{bx - x^2}{a} - x^2g(w, r)$$

One therefore finds profit maximizing level of output by

$$\max_{wrt x} \pi(x) = \frac{bx - x^2}{a} - x^2g(w, r)$$

Look for a critical value of x ; that is, an x_o such that $\pi_x(x_o) = 0$.

$$\frac{\partial \pi(x)}{\partial x} = \frac{b - 2x}{a} - 2xg(w, r)$$

Setting this equal to 0 and solving for x , one obtains

$$\begin{aligned} \frac{b - 2x}{a} - 2xg(w, r) &= 0 \Rightarrow \\ \frac{b - 2x}{a(2x)} - g(w, r) &= 0 \Rightarrow \\ \frac{b}{a2x} - \frac{1}{a} - g(w, r) &= 0 \Rightarrow \\ \frac{b}{2x} - 1 - ag(w, r) &= 0 \Rightarrow \\ \frac{b}{2x} &= 1 + ag(w, r) \Rightarrow \\ \frac{2x}{b} &= \frac{1}{(1 + ag(w, r))} \Rightarrow \\ x^o &= \frac{.5b}{(1 + ag(w, r))} \end{aligned}$$

This might be the monopolist's profit maximizing level of output as a function of the input prices. Check to make sure this critical point is a maximum. Look at

$$\frac{\partial^2 \pi(x)}{\partial x^2} = \frac{-2}{a} - 2g(w, r) < 0 \text{ because } a > 0 \text{ and } g(w, r) > 0$$

Since the second derivative of profits is negative for all values of x , it is negative for $x = x_o$, so the critical point does maximize profits and

$$\begin{aligned} x^s &= x^s(w, r) = \frac{.5b}{(1 + ag(w, r))} \\ &= .5b(1 + ag(w, r))^{-1} \end{aligned}$$

It does not depend on p because p is not exogenous to the monopolistic firm. With respect to predicting what happens to x^s if r increases.

$$\begin{aligned} \frac{\partial x^s}{\partial r} &= \frac{\partial (.5b(1 + ag(w, r))^{-1})}{\partial r} \\ &= -.5b(1 + ag(w, r))^{-2} \left(a \frac{\partial g(w, r)}{\partial r} \right) \\ &= -.5b(1 + ag(w, r))^{-2} (ag_r(w, r)) \\ &= -\frac{.5bag_r(w, r)}{(1 + ag(w, r))^2} < 0 \end{aligned}$$

because by assumption $a, b, g(w, r)$ and $g_r(w, r) > 0$

33. (15 points) Assume a profit maximizing **monopolist**. The demand function for its output is

$$x(p) = b - ap, \text{ where } a, b, > 0$$

The cost function is

$$c(x, w, r) = xg(w, r)$$

where $g(w, r) > 0 \forall w, r > 0$, $g_w(w, r) > 0$ and $g_r(w, r) > 0$. Derive the profit maximizing level of output, x^s , as a function of w and r . That is derive $x^s = x^s(w, r)$. Don't forget to check the second order conditions. Why doesn't the profit maximizing level of output depend on p ? Does this model predict that the profit maximizing level of x will decrease if r increases? Yes or No and prove it.

Answer: The goal is to derive the profit maximizing level of output as a function of the exogenous input prices, w and r . Note that for the monopolist, price is endogenous. What is exogenous is the aggregate demand function for the firm's product. The first step is express the firm's profits as a function of x . Costs are already expressed in unit of x . Total revenue also needs to be expressed in units of x . Total revenue is price times

quantity, but price is a function of quantity. That is, one want to express total revenue as a function of x .

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So

$$\pi(x) = \frac{bx - x^2}{a} - xg(w, r)$$

One therefore finds profit maximizing level of output by

$$\max_{wrt x} \pi(x) = \frac{bx - x^2}{a} - xg(w, r)$$

Look for a critical value of x ; that is, an x_o such that $\pi_x(x_o) = 0$.

$$\frac{\partial \pi(x)}{\partial x} = \frac{b - 2x}{a} - g(w, r)$$

Setting this equal to 0 and solving for x , one obtains

$$\begin{aligned} \frac{b - 2x}{a} &= g(w, r) \Rightarrow \\ \frac{b}{a} - \frac{2x}{a} &= g(w, r) \Rightarrow \\ \frac{b}{a} - g(w, r) &= \frac{2x}{a} \Rightarrow \\ .5(b - ag(w, r)) &= x_o \end{aligned}$$

This might be the monopolist's profit maximizing level of output as a function of the input prices. Check to make sure this critical point is a maximum. Look at

$$\frac{\partial^2 \pi(x)}{\partial x^2} = \frac{-2}{a} < 0 \text{ because } a > 0$$

Since the second derivative of profits is negative for all values of x , it is negative for $x = x_o$, so the critical point does maximize profits and

$$x^s = x^s(w, r) = .5(b - ag(w, r))$$

It does not depend on p because p is not exogenous to the monopolistic firm. With respect to predicting what happens to x^s if r increases.

$$\frac{\partial x^s}{\partial r} = -.5a \frac{\partial g(w, r)}{\partial r} < 0 \text{ because by assumption } g_r(w, r) > 0 \text{ and } a > 0$$

34. Assume a profit maximizing **monopolist**. The demand function for its output is

$$x(p) = b - ap, \text{ where } a, b, > 0$$

The cost function is

$$c(x, w, r) = x^2 g(w, r)$$

where $g(w, r) > 0 \forall w, r > 0$, $g_w(w, r) > 0$ and $g_r(w, r) > 0$. Derive the profit maximizing level of output, x^s , as a function of w and r . That is derive $x^s = x^s(w, r)$. Don't forget to check the second order conditions. Why doesn't the profit maximizing level of output depend on p ? Does this model predict that the profit maximizing level of x will decrease if r increases? Yes or No and prove it.

Answer: The goal is to derive the profit maximizing level of output as a function of the exogenous input prices, w and r . Note that for the monopolist, price is endogenous. What is exogenous is the aggregate demand function for the firm's product. The first step is express the firm's profits as a function of x . Costs are already expressed in unit of x . Total revenue also needs to be expressed in units of x . Total revenue is price terms quantity, but price is a function of quantity. That is, one want to express total revenue as a function of x .

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One therefore finds profit maximizing level of output by

$$\max_{wrt x} \pi(x) = \frac{bx - x^2}{a} - x^2 g(w, r)$$

Look for a critical value of x ; that is, an x_o such that $\pi_x(x_o) = 0$.

$$\frac{\partial \pi(x)}{\partial x} = \frac{b - 2x}{a} - 2xg(w, r)$$

Setting this equal to 0 and solving for x , one obtains

$$\begin{aligned}
 \frac{b-2x}{a} - 2xg(w,r) &= 0 \Rightarrow \\
 \frac{b-2x}{a(2x)} - g(w,r) &= 0 \Rightarrow \\
 \frac{b}{a2x} - \frac{1}{a} - g(w,r) &= 0 \Rightarrow \\
 \frac{b}{2x} - 1 - ag(w,r) &= 0 \Rightarrow \\
 \frac{b}{2x} &= 1 + ag(w,r) \Rightarrow \\
 \frac{2x}{b} &= \frac{1}{(1 + ag(w,r))} \Rightarrow \\
 x^o &= \frac{.5b}{(1 + ag(w,r))}
 \end{aligned}$$

This might be the monopolist's profit maximizing level of output as a function of the input prices. Check to make sure this critical point is a maximum. Look at

$$\frac{\partial^2 \pi(x)}{\partial x^2} = \frac{-2}{a} - 2g(w,r) < 0 \text{ because } a > 0 \text{ and } g(w,r) > 0$$

Since the second derivative of profits is negative for all values of x , it is negative for $x = x^o$, so the critical point does maximize profits and

$$\begin{aligned}
 x^s &= x^s(w,r) = \frac{.5b}{(1 + ag(w,r))} \\
 &= .5b(1 + ag(w,r))^{-1}
 \end{aligned}$$

It does not depend on p because p is not exogenous to the monopolistic firm. With respect to predicting what happens to x^s if r increases.

$$\begin{aligned}
 \frac{\partial x^s}{\partial r} &= \frac{\partial (.5b(1 + ag(w,r))^{-1})}{\partial r} \\
 &= -.5b(1 + ag(w,r))^{-2} \left(a \frac{\partial g(w,r)}{\partial r} \right) \\
 &= -.5b(1 + ag(w,r))^{-2} (ag_r(w,r)) \\
 &= -\frac{.5bag_r(w,r)}{(1 + ag(w,r))^2} < 0
 \end{aligned}$$

because by assumption $a, b, g(w,r)$ and $g_r(w,r) > 0$

35. (15 points) Assume a profit maximizing **monopolist**. The demand function for its output is

$$x(p) = b - ap, \text{ where } a, b, > 0$$

The cost function is

$$c(x, w, r) = xwr$$

where $g(w, r) > 0 \forall w, r > 0$, $g_w(w, r) > 0$ and $g_r(w, r) > 0$. Derive the profit maximizing level of output, x^s , as a function of w and r . That is derive $x^s = x^s(w, r)$. Don't forget to check the second order conditions. Why doesn't the profit maximizing level of output depend on p ? Does this model predict that the profit maximizing level of x will decrease if r increases? Yes or No and prove it.

Answer: The goal is to derive the profit maximizing level of output as a function of the exogenous input prices, w and r . Note that for the monopolist, price is endogenous. What is exogenous is the aggregate demand function for the firm's product. The first step is express the firm's profits as a function of x . Costs are already expressed in unit of x . Total revenue also needs to be expressed in units of x . Total revenue is price terms quantity, but price is a function of quantity. That is, one want to express total revenue as a function of x .

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$$p(x) = \frac{b - x}{a}$$

So,

$$tr(x) = p(x)x = \frac{(b - x)x}{a} = \frac{bx - x^2}{a}$$

So

$$\pi(x) = \frac{bx - x^2}{a} - xwr$$

One therefore finds profit maximizing level of output by

$$\max_{wrt x} \pi(x) = \frac{bx - x^2}{a} - xwr$$

Look for a critical value of x ; that is, an x_o such that $\pi_x(x_o) = 0$.

$$\frac{\partial \pi(x)}{\partial x} = \frac{b - 2x}{a} - wr$$

Setting this equal to 0 and solving for x , one obtains

$$\begin{aligned} \frac{b - 2x}{a} &= wr \Rightarrow \\ \frac{b}{a} - \frac{2x}{a} &= wr \Rightarrow \\ \frac{b}{a} - wr &= \frac{2x}{a} \Rightarrow \\ .5(b - awr) &= x_o \end{aligned}$$

This might be the monopolist's profit maximizing level of output as a function of the input prices. Check to make sure this critical point is a maximum. Look at

$$\frac{\partial^2 \pi(x)}{\partial x^2} = \frac{-2}{a} < 0 \text{ because } a > 0$$

Since the second derivative of profits is negative for all values of x , it is negative for $x = x_o$, so the critical point does maximize profits and

$$x^s = x^s(w, r) = .5(b - awr)$$

It does not depend on p because p is not exogenous to the monopolistic firm. With respect to predicting what happens to x^s if r increases.

$$\frac{\partial x^s}{\partial r} = \frac{\partial(.5(b - awr))}{\partial r} = -0.5aw < 0 \text{ because } a \text{ is positive by assumption}$$

Note that this problem is a special case of question 26, so if you already did question 26, all you need to here is replace $g(w, r)$ wr .

36. Consider econometrics and OLS (ordinary least squares). Assume

$$y_i = \beta x_i + \varepsilon_i \quad i = 1, 2.$$

where β is a parameter and ε_i is a random term that varies from observation to observation. Assume you have two observations in your data set

i	y_i	x_i
1	9	4
2	5	3

OLS finds those estimate of β , $\hat{\beta}$, that minimize the sum of the squared residuals. The squared residual for observation i is

$$(y_i - \hat{\beta}x_i)^2$$

where $\hat{\beta}x_i$ is the predicted value of y_i . Find the OLS estimates of β . What is your best estimate of y_i if $x_i = 10$?

Answer:

$$\min_{wrt \beta} SSR = (9 - \beta 4)^2 + (5 - \beta 3)^2$$

Find the critical point

$$\begin{aligned} \frac{\partial SSR}{\partial \beta} &= -2(9 - \beta 4)4 - 2(5 - \beta 3)3 = -2[(9 - \beta 4)4 + (5 - \beta 3)3] \\ &= -102 + 50\beta \end{aligned}$$

Set this equal to zero and solve for β

$$\begin{aligned} -102 + 50\beta &= 0 \Rightarrow \\ \beta &= \frac{102}{50} \approx 2 \end{aligned}$$

Check the second-order conditions.

$$\frac{\partial^2 SSR}{\partial \beta^2} = 50 > 0$$

So $\hat{\beta} \approx 2$ minimizes the SSR .

37. Consider econometrics and OLS (ordinary least squares). Assume

$$y_i = \beta x_i + \varepsilon_i \quad i = 1, 2.$$

where β is a parameter and ε_i is a random term that varies from observation to observation. Assume you have two observations, $i = 1, 2$, in your data set

i	y_i	x_i
1	10	4
2	6	2

OLS finds those estimate of β , $\hat{\beta}$, that minimize the sum of the squared residuals. The squared residual for observation i is

$$(y_i - \hat{\beta}x_i)^2$$

where $\hat{\beta}x_i$ is the predicted value of y_i . Find the OLS estimates of β . What is your best estimate of y_i if $x_i = 10$? When you are finished with the problem to this point, add a third observation and re-estimate β .

Answer:

$$\min_{wrt \beta} SSR = (10 - \beta 4)^2 + (6 - \beta 2)^2$$

Find the critical point

$$\begin{aligned} \frac{\partial SSR}{\partial \beta} &= -2(10 - \beta 4)4 - 2(6 - \beta 2)2 = -2[(10 - \beta 4)4 + (6 - \beta 2)2] \\ &= -104 + 40\beta \end{aligned}$$

Set this equal to zero and solve for β

$$\begin{aligned} -104 + 40\beta &= 0 \Rightarrow \\ \hat{\beta} &= \frac{104}{40} = 2.6 \end{aligned}$$

Check the second-order conditions.

$$\frac{\partial^2 SSR}{\partial \beta^2} = 40 > 0$$

So $\hat{\beta} = 2.6$ minimizes the SSR .

Check the second-order conditions.

$$\frac{\partial^2 SSR}{\partial \beta^2} = 32 + 18 = 40 > 0$$

38. Consider econometrics and OLS (ordinary least squares). Assume

$$y_i = \alpha + \beta x_i + \varepsilon_i \quad i = 1, 2, 3.$$

where α and β are parameters and ε_i is a random term that varies from observation to observation. Assume you have three observations in your data set

i	y_i	x_i
1	8	1
2	12	2
3	16	3

OLS finds those estimate of α and β , $\hat{\alpha}$ and $\hat{\beta}$, that minimize the sum of the squared residuals. The squared residual for observation i is

$$(y_i - \hat{\alpha} - \hat{\beta}x_i)^2$$

where \hat{y} is the predicted value of y_i . Find the OLS estimates of α and β . What is your best estimate of y_i if $x_i = 10$?

answer:

$$\begin{aligned} ssr(\alpha, \beta) &= (8 - \alpha - \beta)^2 + (12 - \alpha - 2\beta)^2 + (16 - \alpha - 3\beta)^2 \\ &= 464 - 72\alpha - 160\beta + 3\alpha^2 + 12\alpha\beta + 14\beta^2 \end{aligned}$$

We want to minimize this wrt α and β . Look for critical points.

$$\begin{aligned} \frac{d}{d\alpha} [464 - 72\alpha - 160\beta + 3\alpha^2 + 12\alpha\beta + 14\beta^2] \\ = -72 + 6\alpha + 12\beta \end{aligned}$$

$$\begin{aligned} \frac{d}{d\beta} [464 - 72\alpha - 160\beta + 3\alpha^2 + 12\alpha\beta + 14\beta^2] \\ = -160 + 12\alpha + 28\beta \end{aligned}$$

Setting these both equal to 0, there are two equation in to unknowns

$$\begin{aligned} -72 + 6\alpha + 12\beta &= 0 \\ -160 + 12\alpha + 28\beta &= 0 \end{aligned}$$

, Stationary point is: $\{\hat{\beta} = 4 \hat{\alpha} = 4\}$. Should we check the second-order conditions? $\frac{d^2}{d\alpha^2} [464 - 72\alpha - 160\beta + 3\alpha^2 + 12\alpha\beta + 14\beta^2] = 6 > 0$ and $\frac{d^2}{d\beta^2} [464 - 72\alpha - 160\beta + 3\alpha^2 + 12\alpha\beta + 14\beta^2] = 28 > 0$, so we are at a min in the direction of the axis. 4 and 4 are likely minimizing the ssr . We could look further just to make sure. We need to check the sign of

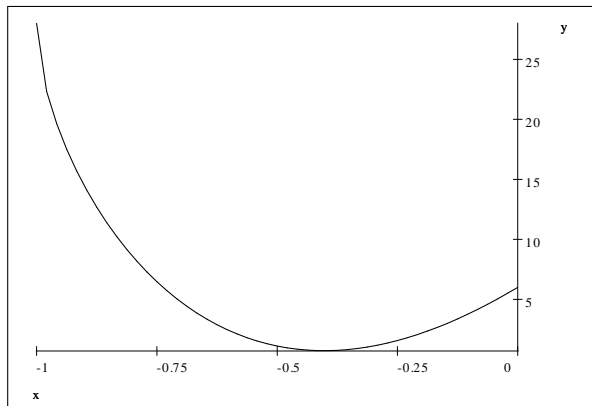
$$ssr_{\alpha\alpha}(\alpha, \beta)v_{\alpha}^2 + ssr_{\beta\beta}(\alpha, \beta)v_{\beta}^2 + 2ssr_{\alpha\beta}(\alpha, \beta)v_{\alpha}v_{\beta}$$

which in this case is

$$6v_\alpha^2 + 28v_\beta^2 + 24v_\alpha v_\beta$$

We need to show this is nonnegative for all feasible values of v . It is obviously positive if v_α and v_β are both positive or both negative. Also if either is equal to plus or minus 1 (in which case the other is 0).

What if one is positive and the other negative? In that case, does $6v_\alpha^2 + 28v_\beta^2 + 24v_\alpha v_\beta \geq 0$? In these directions, the first two terms are always positive and the last one negative. What do we know about v_α and v_β ? $v_\alpha^2 + v_\beta^2 = 1$, Solving for α solution is: $\left\{ v_\alpha = \sqrt{(1 - v_\beta^2)} \right\}, v_\alpha = -\sqrt{(1 - v_\beta^2)}$. Without loss of generality, go with the first one, implying $v_\beta < 0$. Substitute for v_α in $6v_\alpha^2 + 28v_\beta^2 + 24v_\alpha v_\beta$ one get $6(\sqrt{(1 - v_\beta^2)})^2 + 28v_\beta^2 + 24\sqrt{(1 - v_\beta^2)}v_\beta$. Graph it in the range $-1 < v_\beta < 0$



It is always positive, so the second-order conditions for a min hold in every direction. Instead of graphing this function, one could have searched for its minimum value and shown that it is always positive. That is, $\frac{d}{dv_\beta} \left[6(\sqrt{(1 - v_\beta^2)})^2 + 28v_\beta^2 + 24\sqrt{(1 - v_\beta^2)}v_\beta \right] = 4 \frac{11\sqrt{(1 - v_\beta^2)}v_\beta - 12v_\beta^2 + 6}{\sqrt{(1 - v_\beta^2)}} = 0$, Solution is -0.40266 , which the graph confirms is the minimum value.

39. Consider econometrics and OLS (ordinary least squares). Assume

$$y_i = \alpha + \beta x_i + \varepsilon_i \quad i = 1, 2, 3.$$

where α and β are parameters and ε_i is a random term that varies from observation to observation. Assume you have three observations in your data set

i	y_i	x_i
1	8	1
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OLS finds those estimate of α and β , $\hat{\alpha}$ and $\hat{\beta}$, that minimize the sum of the squared residuals. The squared residual for observation i is

$$(y_i - \hat{\alpha} - \hat{\beta}x_i)^2$$

where \hat{y} is the predicted value of y_i . Find the OLS estimates of α and β . What is your best estimate of y_i if $x_i = 10$?

answer:

$$\begin{aligned} ssr(\alpha, \beta) &= (8 - \alpha - \beta)^2 + (12 - \alpha - 2\beta)^2 + (16 - \alpha - 3\beta)^2 \\ &= 464 - 72\alpha - 160\beta + 3\alpha^2 + 12\alpha\beta + 14\beta^2 \end{aligned}$$

We want to minimize this wrt α and β . Look for critical points.

$$\begin{aligned} &\frac{d}{d\alpha} [464 - 72\alpha - 160\beta + 3\alpha^2 + 12\alpha\beta + 14\beta^2] \\ &= -72 + 6\alpha + 12\beta \end{aligned}$$

$$\begin{aligned} &\frac{d}{d\beta} [464 - 72\alpha - 160\beta + 3\alpha^2 + 12\alpha\beta + 14\beta^2] \\ &= -160 + 12\alpha + 28\beta \end{aligned}$$

Setting these both equal to 0, there are two equation in two unknowns

$$\begin{aligned} -72 + 6\alpha + 12\beta &= 0 \\ -160 + 12\alpha + 28\beta &= 0 \end{aligned}$$

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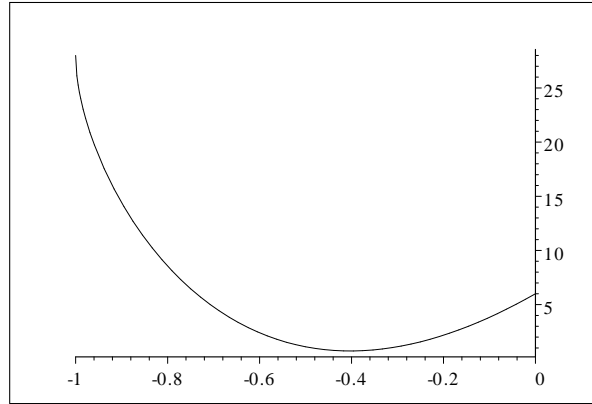
$$ssr_{\alpha\alpha}(\alpha, \beta)v_\alpha^2 + ssr_{\beta\beta}(\alpha, \beta)v_\beta^2 + 2ssr_{\alpha\beta}(\alpha, \beta)v_\alpha v_\beta$$

which in this case is

$$6v_\alpha^2 + 28v_\beta^2 + 24v_\alpha v_\beta$$

We need to show this is nonnegative for all feasible values of v . It is obviously positive if v_α and v_β are both positive or both negative. Also if either is equal to plus or minus 1 (in which case the other is 0).

What if one is positive and the other negative? In that case, does $6v_\alpha^2 + 28v_\beta^2 + 24v_\alpha v_\beta \geq 0$? In these directions, the first two terms are always positive and the last one negative. What do we know about v_α and v_β ? $v_\alpha^2 + v_\beta^2 = 1$, Solving for α solution is: $\left\{ v_\alpha = \sqrt{(1 - v_\beta^2)} \right\}, v_\alpha = -\sqrt{(1 - v_\beta^2)}$. Without loss of generality, go with the first one, implying $v_\beta < 0$. Substitute for v_α in $6v_\alpha^2 + 28v_\beta^2 + 24v_\alpha v_\beta$ one get $6(\sqrt{(1 - v_\beta^2)})^2 + 28v_\beta^2 + 24\sqrt{(1 - v_\beta^2)}v_\beta$. Graph it in the range $-1 < v_\beta < 0$



It is always positive, so the second-order conditions for a min hold in every direction. Instead of graphing this function, one could have searched for its minimum value and shown that it is always positive. That is, $\frac{d}{dv_\beta} \left[6(\sqrt{(1 - v_\beta^2)})^2 + 28v_\beta^2 + 24\sqrt{(1 - v_\beta^2)}v_\beta \right] = 4 \frac{11\sqrt{(1 - v_\beta^2)}v_\beta - 12v_\beta^2 + 6}{\sqrt{(1 - v_\beta^2)}} = 0$, Solution is -0.40266 , which the graph confirms is the minimum value.