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Advanced Microeconomics 7050 Homework 2 Answer Key

Problem 1

First, specify the Pareto optimal efficient allocation conditions by Planner's problem. Then, characterize the competitive market conditions by UMP and PMP.

1) Assumptions

- Two goods: clothing (C) and food (F)
- Household: Ann (A) and Bill (B)
- Inputs: labor (L) and capital (K) for both clothing (C) and food (F) productions.
- Production function is strictly concave.
- Production function and utility function satisfy Inada condition.
- Preferences are monotonic and convex.
- Household owns fractions θ^{hf} of factors.

Consumptions of household (*h=A,B*): $x^h = \{x_C^h, x_F^h\}$

Outputs for good *i*: $y = \{y_C, y_F\}$

1. Solve following planner's problem to characterize Pareto efficient allocations.

$$\max u^A(x_C^A, x_F^A)$$

s.t.

$$\begin{cases} u^B(x_C^B, x_F^B) \geq \bar{u}^B & ...(1-1) \\ f_C(L_C, K_C) \geq y_C & ...(1-2) \\ f_F(L_F, K_F) \geq y_F & ...(1-3) \\ L_C + L_F \geq \bar{L} & ...(1-4) \\ K_C + K_F \geq \bar{K} & ...(1-5) \\ x_C^A + x_C^B \leq y_C & ...(1-6) \\ x_F^A + x_F^B \leq y_F & ...(1-7) \\ x_i^h \geq 0 & ...(1-8) \\ y_i \geq 0 & ...(1-9) \\ L_i \geq 0 & ...(1-10) \\ K_i \geq 0 & ...(1-11) \end{cases}$$

where equation (1-1) is reservation utilities for Bill, (1-2) and (1-3) are production feasibility, and (1-4) and (1-5) are factor market clearing conditions, (1-6) and (1-7) are market clearing conditions for both good clothing and food, and (1-8) to (1-11) are non-negativity conditions.

Because 1) preferences are monotonic and convex and 2) we assume Inada conditions for production function and utility function, the problem admits no corner solutions, so the non-negativity constraints will never bind and can be dropped. Thus, we solve the problem using the Lagrangian.

$$l = u^A(x_C^A, x_F^A) + \lambda(\bar{u}^B - u^B(x_C^B, x_F^B)) + \mu_C(f_C(L_C, K_C) - y_C) + \mu_F(f_F(L_F, K_F) - y_F)$$
$$+\delta_L(\bar{L} - (L_C + L_F)) + \delta_K(\bar{K} - (K_C + K_F)) + \gamma_C(y_C - (x_C^A + x_C^B)) + \gamma_F(y_F - (x_F^A + x_F^B))$$

We will have following first order conditions:

$$x_C^A: \frac{\partial l}{\partial x_C^A} = \frac{\partial u^A}{\partial x_C^A} - \gamma_C = 0 ...(A-1)$$

$$x_F^A: \frac{\partial l}{\partial x_F^A} = \frac{\partial u^A}{\partial x_F^A} - \gamma_F = 0 ...(A-2)$$

$$x_C^B: \frac{\partial l}{\partial x_C^B} = \lambda \frac{\partial u^B}{\partial x_C^B} - \gamma_C = 0 ...(B-1)$$

$$x_F^B: \frac{\partial l}{\partial x_F^B} = \lambda \frac{\partial u^B}{\partial x_F^B} - \gamma_F = 0 ...(B-2)$$

$$L_C: \frac{\partial l}{\partial L_C} = \mu_C \frac{\partial f_C}{\partial L_C} - \delta_L = 0 ...(C-1)$$

$$L_F: \frac{\partial l}{\partial L_F} = \mu_F \frac{\partial f_F}{\partial L_F} - \delta_L = 0 ...(C-2)$$

$$K_C: \frac{\partial l}{\partial K_C} = \mu_C \frac{\partial f_C}{\partial K_C} - \delta_K = 0 ...(D-1)$$

$$K_F: \frac{\partial l}{\partial K_F} = \mu_F \frac{\partial f_F}{\partial K_F} - \delta_K = 0 ...(D-2)$$

$$y_C: \frac{\partial l}{\partial y_C} = -\mu_C + \gamma_C = 0 ...(E-1)$$

$$y_F: \frac{\partial l}{\partial y_F} = -\mu_F + \gamma_F = 0 ...(E-2)$$

From (A)'s and (B)'s, we have following condition.

$$\frac{\partial u^A}{\partial x_C^A} \Big/ \frac{\partial u^A}{\partial x_F^A} = \frac{\partial u^B}{\partial x_C^B} \Big/ \frac{\partial u^B}{\partial x_F^B} \Rightarrow MRS^A = MRS^B \ldots\text{(Condition 1)}$$

From (C)'s and (D)'s, we have following condition.

$$\frac{\partial f_C}{\partial L_C} \bigg/ \frac{\partial f_C}{\partial K_C} = \frac{\partial f_F}{\partial L_F} \bigg/ \frac{\partial f_F}{\partial K_F} \Rightarrow \frac{MPL_C}{MPK_C} = \frac{MPL_F}{MPK_F} \Rightarrow MRT_C = MRT_F \ldots\text{(Condition 2)}$$

From (E)'s, (A)'s, (C)'s, and (D)'s, we have following condition.

$$\frac{\partial u^A}{\partial x_C^A} \bigg/ \frac{\partial u^A}{\partial x_F^A} = \frac{\partial f_C}{\partial L_C} \bigg/ \frac{\partial f_C}{\partial K_C} \Rightarrow MRS = MRT \ldots\text{(Condition 3)}$$

We need (Condition 1) to (Condition 3) to characterize Pareto efficient allocations.

In addition, constraints (1-1) – (1-7) hold as equalities.

2. Competitive outcomes

1) Households face given market prices and each household maximizes utility.

$$\max u^h(x_C^h, x_F^h)$$
$$\text{s.t. } p_C x_C^h + p_F x_F^h \le wL\theta^{hL} + rK\theta^{hK} \ldots(2)$$

First order conditions will be:

$$\frac{\partial u^h}{\partial x_C^h} \bigg/ \frac{\partial u^h}{\partial x_F^h} = p_C / p_F \ldots(3)$$

We have the condition (3) for both Ann and Bill.

Thus, we have $\frac{\partial u^A}{\partial x_C^A} \bigg/ \frac{\partial u^A}{\partial x_F^A} = \frac{\partial u^B}{\partial x_C^B} \bigg/ \frac{\partial u^B}{\partial x_F^B} \Rightarrow MRS_A = MRS_B$. It is same as (Condition 1).

2) Firms producing clothing and food faces given factor prices and market prices of goods and maximize their profits.

$$\max\ p_i y_i - (wL_i + rK_i)$$
$$\text{s.t. } f_i(L_i, K_i) \ge y_i \ldots(4)$$

Solve this problem by Lagrange,

$$l = p_i y_i - (wL_i + rK_i) + \lambda_i [f_i(L_i, K_i) - y_i]$$

We will have following first order conditions:

$$y_i:\ p_i - \lambda_i = 0 \ldots(5\text{-}1)$$

$$L_i:\ -w + \lambda_i \frac{\partial f_i}{\partial L_i} = 0 \ldots(5\text{-}2)$$

$K_i:\ -r+\lambda_i \dfrac{\partial f_i}{\partial K_i}=0 \ldots$(5-3)

We have the conditions (5-1) to (5-3) for clothing and food.

From (5-2) and (5-2) for both goods, $\dfrac{\partial f_C}{\partial L_C}\Big/\dfrac{\partial f_C}{\partial K_C}=\dfrac{\partial f_F}{\partial L_F}\Big/\dfrac{\partial f_F}{\partial K_F}\Rightarrow MRT_C=MRT_F$.

It is same as (Condition 2).

From (5-1), (5-2), and (5-2) as well as (3), we have

$\dfrac{\partial u^A}{\partial x_C^A}\Big/\dfrac{\partial u^A}{\partial x_F^A}=\dfrac{\partial f_C}{\partial L_C}\Big/\dfrac{\partial f_C}{\partial K_C}\Rightarrow MRS=MRT$ which is (Condition 3).

Market clearing conditions and constraints in PMP and UMP imply conditions (1-2) — (1-7) which hold as equalities. In equilibrium, the level of individual utility is determined by the value of initial endowments under market prices.

From above, competitive outcomes allocations are identical to Pareto optimal allocations. Thus, the fist welfare theorem holds.

Problem 3

First, we need to solve the utility maximizing problem for each consumer.

1) Consumer 1: $\max\limits_{x_{11},x_{21}}[x_{11}-\frac{1}{8}(x_{21})^{-8}]$ s.t $P_1x_{11}+P_2x_{21}\le P_1\cdot 2+P_2\cdot r$

2) Consumer 2: $\max\limits_{x_{12},x_{22}}[x_{22}-\frac{1}{8}(x_{12})^{-8}]$ s.t $P_1x_{12}+P_2x_{22}\le P_1\cdot r+P_2\cdot 2$

Solving these problems, then we have offer curves.

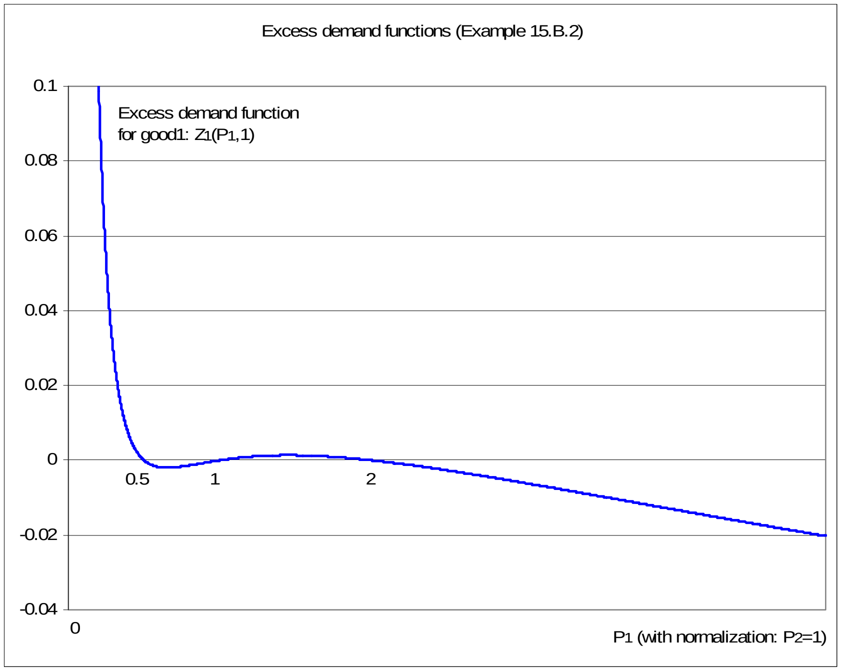
3) Consumer 1: $OC_1(P_1,P_2)=\left[2+r(\frac{P_2}{P_1})-(\frac{P_2}{P_1})^{\frac{8}{9}},(\frac{P_1}{P_2})^{\frac{1}{9}}\right]$

4) Consumer 2: $OC_2(P_1,P_2)=\left[(\frac{P_2}{P_1})^{\frac{1}{9}},2+r(\frac{P_1}{P_2})-(\frac{P_1}{P_2})^{\frac{8}{9}}\right]$

In this exchange economy where supply is fixed, we can have excess demand function as total demand minus total supply. (It is sufficient to solve the excess demand function for one good because of Walras' law).

5) Excess demand for good 1: $Z_1(\frac{P_2}{P_1})=2+r(\frac{P_2}{P_1})-(\frac{P_2}{P_1})^{\frac{8}{9}}+(\frac{P_2}{P_1})^{\frac{1}{9}}-(2+r)$

We can draw the following graph (whatever software you can use…) and you can see there are 3 answers solutions: 0.5, 1, and 2 as mentioned in MWG. Also see p590 Figure 17.D.1). I normalize P_2=1.



From the definition 17.D.2 (p.592), index p=$(-1)^{L-1}$sign|Dz(p)|. Because L=2, then |Dz(p)| is just the derivative of the z(p). With normalization of P_2, we have the following:

$Dz(P_1) = -rP_1^{-2} + (8/9)P_1^{-17/9} - (1/9)P_1^{-10/9}$.

For each price, we are going to calculate the index.
Index (P_1=0.5)=$(-1)^1$sign|Dz(P_1=0.5)|=(-1)*sign|-0.03477|=1,
Index (P_1=1)=$(-1)^1$sign|Dz(P_1=1)|=(-1)*sign|0.00608|=-1,
Index (P_1=2)=$(-1)^1$sign|Dz(P_1=2)|=(-1)*sign|-0.00434|=1.

Using "Proposition 17.D.2 (Index theorem)" and results above, $\sum_{\{P_1:Z(P_1)=0,P_2=1\}} indexP_1 = 1$.

We can conclude that it is a regular economy.