ECON 7050
HW1

September 20, 2008

Problem 1 *Consider the Edgeworth box economy discussed in class. There are 10 pheasants and 10 buckets of apples, Adam and Eve have preferences over their own consumption bundles* $(x_{1i}, x_{2i}) \in \mathbb{R}^2_+$.

Characterize the set of Pareto efficient allocations (PEA) for the economy, where preferences are represented by

1. $u^i(x_{1i}, x_{2i}) = \sqrt{x_{1i} x_{2i}},\ i \in \{A, E\}$.

$$\begin{aligned} & \max_{x_{1A}, x_{2A}, x_{1B}, x_{2B}} u^A(x_{1A}, x_{2A}) \\ & s.t. \\ u^B(x_{1B}, x_{2B}) \geq {} & \bar{u} \end{aligned}$$

Solution 2 *Feasibility:* $\sum_{i \in \{A,E\}} x_{1i} = 10; \sum_{i \in \{A,E\}} x_{2i} = 10$

Optimality conditions. The feasible set is convex (as utilities are, and feasibility constraints are linear). Let α, β, γ *be the multipliers associated with the 3 constraints (non-negativity constraints are dropped for the moment): then*

$$Du^A(x_{1A}, x_{2A}) = (\beta, \gamma) \quad (1)$$
$$\alpha Du^B(x_{1B}, x_{2B}) = (\beta, \gamma) \quad (2)$$

This implies

$$\frac{x_{2A}}{x_{1A}} = \frac{x_{2B}}{x_{1B}}$$

which together with the feasibility constraints gives the "diagonal" in the Edgeworth box: $x_{1A} = x_{2A}$,*so the set of PEA is a subset of* $\mathbb{R}^4_+$ *such that* $\left\{ \sum_{i \in \{A,E\}} x_{1i} = 10; \sum_{i \in \{A,E\}} x_{2i} = 10, x_{1A} = x_{2A} \right\}$.

2. $u^A(x_{1A}, x_{2A}) = x_{1A}^{2/3} x_{2A}^{1/3}$ *and* $u^E(x_{1E}, x_{2E}) = x_{1E}^{1/3} x_{2E}^{2/3}$

Solution 3 *Similar to the above, conditions* $(1, 2)$ *imply*

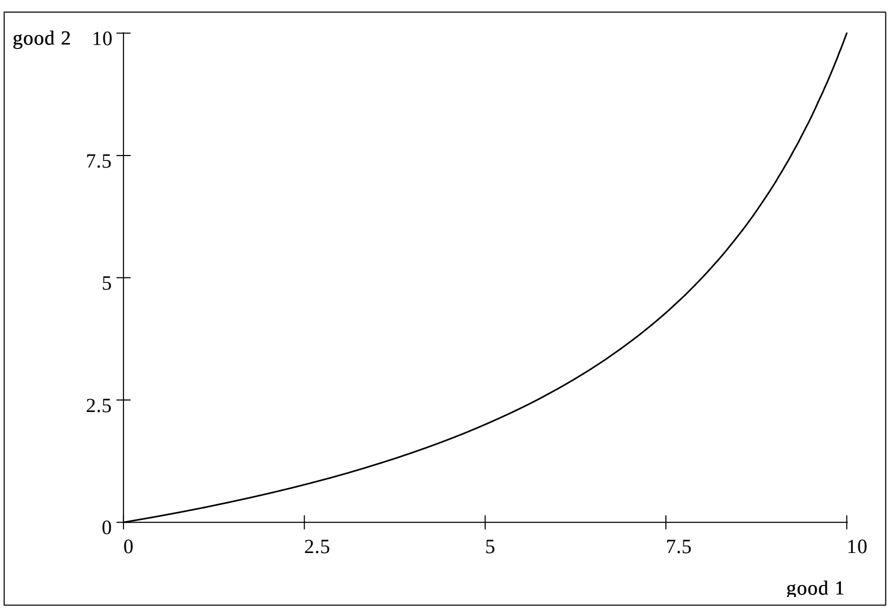
$$\frac{2x_{2A}}{x_{1A}} = \frac{x_{2B}}{2x_{1B}}$$

using feasibility,

$$\frac{2x_{2A}}{x_{1A}} = \frac{10 - x_{2A}}{2(10 - x_{1A})}$$

so, $x_{1A} = 40\frac{x_{2A}}{3x_{2A}+10}$ *and this along with the feasibility conditions describes the set of POA, a subset of* $\mathbb{R}^4_+$

Depict this set in an Edgeworth box for each of the cases.$40\frac{y}{3y+10}$



Can you use the above algorithm for the following specifications of preferences:

3. $u^E(x_{1E}, x_{2E}) = ax_{1E} + x_{2E}$, $u^A(x_{1A}, x_{2A}) = bx_{1A} + x_{2A}$ *with* $a, b > 0$

   Solution 4 *It is possible to use the same procedure, only one has to be aware that the non-negativity constraints can be binding (the max problem is linear). You could also argue in the following way:*

   If $a = b$ *then any feasible allocation is Pareto optimal. Indeed, pick any feasible* $x : u^E(x_{1E}, x_{2E}) = k, u^A(x_{1A}, x_{2A}) = u^A(10 - x_{1E}, 10 - x_{2E}) = v$.

   If we now take $y_E = (y_{1E}, y_{2E})$ *such that* $u(y_{1E}, y_{2E}) = (a, 1)(y_{1E}, y_{2E}) > k$, *then* $(a, 1)((y_{1E}, y_{2E}) - (x_{1E}, x_{2E})) > 0$, *so* $(a, 1)((y_{1E} - x_{1E}, y_{2E} - x_{2E})) > 0$, *but then, given* $10 - y_{1E} - (10 - x_{1E}) = x_E - y_E$,*we have*

   $$(b, 1)(((10 - y_{1E}) - (10 + x_{1E}), (10 - y_{2E}) - (10 - x_{2E}))) < 0$$

   so $u^A(10 - x_{1E}, 10 - x_{2E}) = v < u^A(10 - y_{1E}, 10 - y_{2E})$.

   Now assume $a > b$.*All allocations in which all of the good is given to Eve and some positive amount of good 2 as well* $(x_{1E}, x_{2E}) = (10, x_{2E})$ *with* $x_{2E} > 0$*: if not, the benevolent designer can always transfer a unit of the first good to Eve from Adam and give back to Adam* $a > b$ *units of the second good (from Eve). Also* $(x_{1E}, x_{2E}) = (x_{1E}, 0)$ *with the rest given to Adam is PO, given* $x_{1E} < 10$, *for the same reason as above.*

   Case $a < b$ *is similar.*

4. $u^E(x_{1E}, x_{2E}) = \min\{ax_{1E}, x_{2E}\}$, $u^A(x_{1A}, x_{2A}) = \min\{bx_{1A}, x_{2A}\}$ *with* $a, b > 0$

Solution 5 *Here we can not use the optimization problem: the economy is not smooth, utility functions are not always differentiable.*

If $a = b = 1$,*then the diagonal:* $x_{1E} = x_{2E}$ *is optimal: to improve the well-being of one is to increase allocations of both goods to this individual, thus reducing the well-being of other. Any other allocation is not optimal: if, say,* $x_{1E} > x_{2E}$ *then* $x_{1A} < x_{2A}$ *(given the total amount of each good is 10) and so by transfering some first good from Eve will not affect her utility while increasing the utility of Adam.*

If $a > 1$ and $b > 1$ or $a < 1$ and $b < 1$ then any allocation between the loci $ax_{1E} = x_{2E}$ and $bx_{1A} = x_{2A}$ is PO. Indeed, say $a > 1$ and $b > 1$: pick $x : ax_{1E} = x_{2E}$ then Adam's utility is $\min\{b(10 - x_{1E}), 10 - x_{2E}\} = \min\{b(10 - x_{1E}), 10 - ax_{1E}\} = 10 - ax_{1E}$, with the latter equality because $a > 1$ and $b > 1$. So, both will have the same utility when the "excess" of the first good, i.e., the difference $b(10 - x_{1E}) - (10 - ax_{1E})$ is distributed in any way between the agents. Transfering the second good, however, will necessarily decrease utility of one individual and increase the other.

If a and b are on different sides of unity, say, $a > 1 > b$ then the lines $ax_{1E} = x_{2E}$ and $bx_{1A} = x_{2A}$ intersect (it is easy to show), at, say $\bar{x}$, and then the POA lie between the two lines. Indeed, at $\bar{x}$, $a\bar{x}_{1E} = \bar{x}_{2E}$ and $b(10 - \bar{x}_{1E}) = 10 - a\bar{x}_{1E}$, so for $x_{1E} > \bar{x}_{1E} = \frac{10(1-b)}{a-b}$, we have $u^A(10 - x_{1E}, 10 - x_{2E}) = 10 - ax_{1E}$, redistributing the "excess" of good 1 is welfare neutral (as in the previous case) and if $x_{1E} < \bar{x}_{1E} = \frac{10(1-b)}{a-b}$, $u^A(10 - x_{1E}, 10 - x_{2E}) = b(10 - x_{1E})$, and then the excess of the second good $10 - ax_{1E} - b(10 - x_{1E}) = bx_E - ax_E - 10b + 10$ can be redistributed without changing the utilities of the agents.

Problem 6 *Characterize (in terms of initial endowments) competitive equilibria, if exist, of the Edgeworth box economy where Adam initially has $\omega_{1A} > 0$ pheasants (and no apples), and Eve has $\omega_{2E} > 0$ apples (and no pheasants) and where*

1. $u^i(x_{1i}, x_{2i}) = \sqrt{x_{1i}x_{2i}}$, $i \in \{A, E\}$

   Consumer demand for strictly positive prices is

   $$(x_{1A}, x_{2A}) = \left(\frac{\omega_{1A}}{2}, \frac{p_1\omega_{1A}}{2p_2}\right), (x_{1E}, x_{2E}) = \left(\frac{p_2\omega_{2E}}{2p_1}, \frac{\omega_{2E}}{2}\right).$$

   Market clearing: $\frac{\omega_{1A}}{2} + \frac{p_2\omega_{2E}}{2p_1} = \omega_{1A}$, *so* $\frac{p_2\omega_{2E}}{p_1} = \omega_{1A}$ *and therefore* $\frac{p_2}{p_1} = \frac{\omega_{1A}}{\omega_{2E}}$. *Note the price of a scarcer good is higher.*

2. $u^A(x_{1A}, x_{2A}) = x_{1A}$, $u^E(x_{1E}, x_{2E}) = x_{1E}x_{2E}$.

   Demand for strictly positive prices: $(x_{1A}, x_{2A}) = (\omega_{1A}, 0)$ *and* $(x_{1E}, x_{2E}) = \left(\frac{p_2\omega_{2E}}{2p_1}, \frac{\omega_{2E}}{2}\right)$. *Then the second market will never clear, excess supply. If $p_1 = 0$, then the first market will have excess demand (because of Eve). The only candidate for equilibrium then is with $p_2 = 0$, $p_1 > 0$.*

In this case, Adam's demand is the same, $(x_{1A}, x_{2A}) = (\omega_{1A}, 0)$, *and Eve's demand is* $(0, x_{2E})$ *for any non-negative* x_{2E}. *Thus, the initial allocation is an equilibrium supported by prices* $p_2 = 0$, $p_1 > 0$.

Problem 7 *Write down a statement and a complete proof of the First welfare theorem for an exchange economy with I agents and L goods. Make sure that you indicate how each of the assumptions you make is used. (See MWG).*

Problem 8 *Find a mistake in the following ('fake') proof the statement that "a Walrasian equilibrium exists in any pure exchange economy in which* $\sum_i \omega_i >> 0$ *and every consumer has continuous, strictly convex, and strongly monotone preferences".*

"Proof": use propositions 17.B.2 and 17.C.2.

The problem with this argument is that it does not cover the case of prices $p \neq 0$ *such that for some good* l $p_l = 0$,*in particular, the first proposition says nothing about behavior of the excess demand for such prices, while the second proposition requires the excess demand to be continuous on the boundaries of the simplex.*