ECON 7050
HW on Contract Theory

due Dec 3

Problem 1 *Consider the monopolist problem studied in class. Assume utility of the buyer is* $u_\theta(x,t) = \theta\sqrt{x} - t$*, and the marginal cost is constant at* c*.*

Assume the monopolist can observe the type of the customer (θ) *and can successfully discriminate across consumers. What bundle* (q,t) *will the monopolist offer to each customer? What are the profits of the monopolist?*

Solution 2 *For every consumer of type* θ*, the monopolist solves*

$$\max_{x,t} t - cx$$
$$s.t. \theta\sqrt{x} - t \geq 0$$

The constraint is binding, $\max_x \theta\sqrt{x} - cx$ *implies* $\theta/(2\sqrt{x}) = c$*, hence* $x = \frac{1}{4}\frac{\theta^2}{c^2}$ *and so* $t = \theta\sqrt{x}\big|_{x=\frac{1}{4}\frac{\theta^2}{c^2}} = \frac{1}{2}\frac{\theta^2}{c}$.

Profits for each θ *are* $\theta\sqrt{x} - cx\big|_{x=\frac{1}{4}\frac{\theta^2}{c^2}} = \frac{1}{4}\frac{\theta^2}{c}$.

Problem 3 *Assume the type is not observable and the monopolist believes that* θ *is distributed uniformly on* $[a,b]$, $b > a > 0$*. What is the optimal menu of contracts that the monopolist will offer in this case?*

Solution 4 *By the revelation principle it is sufficient to look at the direct* *revelation mechanisms.*

$$\begin{aligned} & \max_{t(\theta),x(\theta)} \frac{1}{b-a}\int_a^b (t(\theta) - cx(\theta))d\theta \\ & s.t. \\ & \theta\sqrt{x(\theta)} - t(\theta) \geq 0 \\ \theta & \in \arg\max_{\hat{\theta}} \theta\sqrt{x\left(\hat{\theta}\right)} - t\left(\hat{\theta}\right) \forall \theta, \hat{\theta} \end{aligned}$$

Using the Mirlees approach, assuming monotonicity of $x(\cdot)$, *the expected utility of type* θ *revealing his type is* $U(\theta) = U(a) + \int_a^\theta \sqrt{x(\theta)}d\theta$, *and given the IR for the lowest type is binding,* $U(a) = 0$. *Hence* $t(\theta) = \theta\sqrt{x(\theta)} - \int_a^\theta \sqrt{x(s)}ds$. *Now we solve the relaxed problem (omitting the monotonicity constraint):* $\max_{x(\theta)} \frac{1}{b-a}\int_a^b (\theta\sqrt{x(\theta)} - \int_a^\theta \sqrt{x(s)}ds - cx(\theta))d\theta$.

Integrating by parts,

$$\begin{aligned} \frac{1}{b-a}\int_a^b \left(\int_a^\theta \sqrt{x(s)}ds\right) d(\theta) &= \frac{\theta-a}{b-a}\int_a^\theta \sqrt{x(s)}ds|_a^b - \int_a^b \sqrt{x(\theta)}\frac{\theta-a}{b-a}d\theta \\ &= \int_a^b \sqrt{x(\theta)}\left(1 - \frac{\theta-a}{b-a}\right) d\theta \end{aligned}$$

so the problem of the monopolist is

$$\max_{x(\theta)} \int_a^b \left(\frac{\theta\sqrt{x(\theta)}}{b-a} - \frac{cx(\theta)}{b-a} - \sqrt{x(\theta)}\left(1 - \frac{\theta-a}{b-a}\right)\right)d\theta$$

FOC: for almost all θ

$$\left(\frac{\theta}{2\sqrt{x(\theta)}} - c\right)\frac{1}{b-a} - \left(1 - \frac{\theta-a}{b-a}\right)\frac{1}{2\sqrt{x(\theta)}} = 0$$

which yields $(\theta/(2\sqrt{x}) - c)\frac{1}{b-a} - \left(1 - \frac{\theta-a}{b-a}\right)\frac{1}{2\sqrt{x}} = 0$, *Solution is:* $x(\theta) = \frac{1}{4}\frac{(2\theta-b)^2}{c^2}$ *and so* $t(\theta) = \theta\sqrt{x} - \int_a^\theta \sqrt{x}ds|_{x=\frac{1}{4}\frac{(2\theta-b)^2}{c^2}} = \frac{a}{2}\frac{(2\theta-b)}{c}$. *Note that monotonicity of* x *holds, so the solution of the relaxed problem is also a solution to the original problem.*

Problem 5 *Formulate and prove the revelation principle.*

Solution 6 *Let Θ be the set of agents (who differ by type) with preferences over outcomes X. Let $M = \times_{\theta\in\Theta} M_\theta$ be the set of actions (messages).*

Definition 7 *A mechanism is M and a map $\varpi : M \to X$*

Agents observe the mechanism and choose the best action (message):

$$m^*_\theta \in \arg\max_{m\in M_\theta} u_\theta \left(\varpi \left(m, m^*_{-\theta}\right)\right)$$

Let $m^* = (m^*_\theta)_{\theta\in\Theta}$ be the equilibrium profile of actions (messages).

Definition 8 *A mechanism (M, ϖ) implements an allocation $x_\varpi \in X$ if*

$$x_\varpi = \varpi (m^*)$$

Definition 9 *Direct revelation mechanism is $(\Theta, \tilde{\varpi})$ such that $\tilde{\varpi} : \Theta \to X$.*

Proposition 10 *(REVELATION PRINCIPLE)*

Any allocation x_ϖ implementable via mechanism ϖ, can be implemented by a direct revelation mechanism, $\tilde{\varpi}$

Proof. Let $\tilde{\varpi}(\theta) = \varpi(m^*_\theta)$ for every θ. ■

Problem 11 *Section 7.2.2 of the textbook. Suppose buyer's value and seller's cost are distributed uniformly on $[1, 50]$ and $[0, 2]$ correspondingly. Calculate the highest level of joint welfare, $U_1(2) + U_2(1)$, that allows the benevolent mediator to implement efficient trading.*

Solution 12 *Following the proof of the M-S theorem, if the incentive constraints are satisfied and efficient trade is imposed, we can express $U_1(2) + U_2(1)$ as*

$$-\int_1^2 F_1(v)\,(1 - F_2(v))\,dv.$$

Given that $F_1(v) = \frac{1}{2}v$ and $F_2(v) = (v-1)/2$, we get

$$-\int_1^2 \frac{1}{2} v\,(1 - (v-1)/2)\,dv = -\frac{13}{24}.$$

In other words, if the benevolent mediator is allowed to 'soften' the IR constraint, so that the interim utility of any type can fall sufficiently below zero, efficient trade can implemented. The highest low bound on individual utilities to always implement efficient trade has to satisfy our condition, $U_1(2) + U_2(1) \geq -\frac{13}{24}$, so, for example, if IR reads, $U_i(t) \geq -\frac{13}{48}$, $i = 1, 2$, efficient trade can be implemented (is IC).