ECON 7050
HW 4
Due Wednesday November 5

November 10, 2008

Problem 1 *MWG 6.C.17*

Problem 2 *Consider and exchange Arrow-Debreu economy with two risk averse individuals (whose preferences over lotteries satisfy the vNM axioms), one commodity and two states of the world. Assume the individual endowments are* $\omega_1 = (1, 0)$ *and* $\omega_2 = (0, 1)$ *(as in Example 19.C.1) and the perceived likelihood of the two states is commonly shared, so that both believe that state 1 happens with probability* $\pi_1 > 0$.

1. *Show that in any competitive equilibrium individual consumption is constant across states ('complete risk sharing'). Will this be true if the agents' beliefs differ, so that the probability that* 1 *attaches to state 1 is higher than that of the second agent,* $\pi_{11} > \pi_{12}$*?*

   Solution 3 *Assume the utility of each individual is differentiable. In a competitive equilibrium consumers are choosing the best affordable bundle, so each chooses the optimal consumption combination (for each state)*

$$\max_{x_{1i}, x_{2i}} \pi_{1i} u_i (x_{1i}) + (1 - \pi_{1i}) u_i (x_{2i})$$
$$s.t.\ p_1 x_{1i} + p_2 x_{2i} = p\omega_i$$

   Optimality conditions then require

$$\frac{u_i^{'} (x_{1i})}{u_i^{'} \left(\frac{1}{p_2} (p\omega_i - p_1 x_{1i}) \right)} = \frac{p_1 (1 - \pi_{1i})}{p_2 \pi_{1i}} \quad (1)$$

   Let

$$\phi_i (z) = \frac{u_i^{'} (z)}{u_i^{'} \left(\frac{1}{p_2} (p\omega_i - p_1 z) \right)}$$

Given concavity of u_i, ϕ_i is decreasing. If the two agents have the same beliefs, so that $\pi_{1i} = \pi_1$, then

$$\phi_1(x_{11}) = \phi_2(x_{12})$$

Remains to show that in this case $\phi_1(x_{11}) = 1$. Assume, to the contrary, $\phi_1(x_{11}) > 1$. This implies that $x_{11} + x_{12} < x_{21} + x_{22}$, which contradicts the market clearing,

$$x_{11} + x_{12} = 1 = x_{21} + x_{22}$$

Similarly, it can not happen in an equilibrium that $\phi_1(x_{11}) = \phi_2(x_{12}) < 1$.

Now, if $\pi_{11} > \pi_{12}$ then by condition (1) and def. of ϕ_i, we get $\phi_1(x_{11}) < \phi_2(x_{12})$ and hence $x_{11} > x_{12}$. The intuition is simple: the first consumer thinks the first state is more likely and plans to consume more in that state.

2. *Show that if $\pi_1 > \pi_2$ then the first individual should consume more in an Arrow-Debreu equilibrium than the second individual.*

   Now we are back in the case when the beliefs of the two consumers are the same, and both agree the first state is more likely. Hence, by the above $\phi_1(x_{11}) = \phi_2(x_{12})$ so that both insure completely: $x_{1i} = x_{2i}$. But this also implies $\frac{p_1(1-\pi_1)}{p_2\pi_1} = 1$, or $\frac{p_1}{p_2} = \frac{\pi_1}{\pi_2} > 1$ and the conclusion follows from the budget constraints (consumer one is "more wealthy").

Problem 4 *19.C.2*

Problem 5 *Consider and exchange Arrow-Debreu economy with two individuals with identical preferences, $U(x_i) = \pi_1 \ln(x_{11i}x_{21i}) + (1-\pi_1)\ln(x_{12i}x_{22i})$, two commodities and two states of the world. Assume the individual endowments are $\omega_1 = (1,1,0,0)$ and $\omega_2 = (0,0,2,2)$, so that the first individual has all the goods in the economy if state 1 occurs, while the second individual has it all, if state 2 happens. Assume also the perceived likelihood of the two states is commonly shared, so that both believe that state 1 happens with probability $\pi_1 > 0$.*

1. *Compute Arrow-Debreu equilibrium.*

   Solution 6 *Consumers choice.*

$$\max_{x_i} \pi_1 \ln(x_{11i}x_{21i}) + (1-\pi_1)\ln(x_{12i}x_{22i})$$
$$s.t.\ px_i = p\omega_i$$

   FOC (for the interior solution)

$$\begin{aligned} DU(x_i) &= \lambda p \\ \left(\frac{\pi_1}{x_{11i}}, \frac{\pi_1}{x_{21i}}, \frac{1-\pi_1}{x_{12i}}, \frac{1-\pi_1}{x_{22i}}\right) &= \lambda p \end{aligned}$$

So for $s = 1, 2$ *and* $i = 1, 2$

$$\begin{aligned} \frac{x_{2si}}{x_{1si}} &= \frac{p_{1s}}{p_{2s}} \\ \frac{x_{12i}}{x_{11i}} &= \frac{(1-\pi_1)\,p_{11}}{\pi_1 p_{12}} \end{aligned}$$

Substitute these conditions

$$\begin{aligned} x_{21i} &= \frac{p_{11}}{p_{21}} x_{11i} \\ x_{12i} &= \frac{(1-\pi_1)\,p_{11}}{\pi_1 p_{12}} x_{11i} \\ x_{22i} &= \frac{p_{12}}{p_{22}} x_{12i} = \frac{(1-\pi_1)\,p_{11}}{\pi_1 p_{22}} x_{11i} \end{aligned}$$

into the budget constraints

$$\begin{aligned} p_{11}x_{111} + p_{21}\frac{p_{11}}{p_{21}}x_{111} + p_{12}\frac{(1-\pi_1)\,p_{11}}{\pi_1 p_{12}}x_{111} + p_{22}\frac{(1-\pi_1)\,p_{11}}{p_{22}\pi_1}x_{111} &= p_{11} + p_{21} \\ p_{11}x_{112} + p_{21}\frac{p_{11}}{p_{21}}x_{112} + p_{12}\frac{(1-\pi_1)\,p_{11}}{\pi_1 p_{12}}x_{112} + p_{22}\frac{(1-\pi_1)\,p_{11}}{p_{22}\pi_1}x_{112} &= 2\,(p_{12} + p_{22}) \end{aligned}$$

$$\begin{aligned} 2p_{11}x_{111} + 2\frac{1-\pi_1}{\pi_1}p_{11}x_{111} &= p_{11} + p_{21} \\ p_{11}x_{112} + \frac{1-\pi_1}{\pi_1}p_{11}x_{112} &= p_{12} + p_{22} \end{aligned}$$

to get individual demand:

$$\begin{aligned} x_{111} &= \pi_1\frac{p_{11} + p_{21}}{2p_{11}} \\ x_{112} &= \pi_1\frac{p_{12} + p_{22}}{p_{11}} \end{aligned}$$

Hence

$$\begin{aligned} x_{211} &= \pi_1\frac{p_{11} + p_{21}}{2p_{21}},\ x_{212} = \pi_1\frac{p_{12} + p_{22}}{p_{21}} \\ x_{121} &= (1-\pi_1)\frac{p_{11} + p_{21}}{2p_{12}},\ x_{122} = (1-\pi_1)\frac{p_{12} + p_{22}}{p_{12}} \\ x_{221} &= (1-\pi_1)\frac{p_{11} + p_{21}}{2p_{22}},\ x_{222} = (1-\pi_1)\frac{p_{12} + p_{22}}{p_{22}} \end{aligned}$$

Market clearing:

$$\begin{aligned}
\pi_1 \frac{p_{11}+p_{21}}{2p_{11}} + \pi_1 \frac{p_{12}+p_{22}}{p_{11}} &= 1 \\
\pi_1 \frac{p_{11}+p_{21}}{2p_{21}} + \pi_1 \frac{p_{12}+p_{22}}{p_{21}} &= 1 \\
(1-\pi_1) \frac{p_{11}+p_{21}}{2p_{12}} + (1-\pi_1) \frac{p_{12}+p_{22}}{p_{12}} &= 2 \\
(1-\pi_1) \frac{p_{11}+p_{21}}{2p_{22}} + (1-\pi_1) \frac{p_{12}+p_{22}}{p_{22}} &= 2
\end{aligned}$$

Clearly, given the first and the third equations have unique solutions in p_{11} and p_{12} correspondingly, it has to be that then $p_{11} = p_{21}$ and $p_{12} = p_{22}$.

Then we have to solve only two equations:

$$\begin{aligned}
\pi_1 + 2\pi_1 \frac{p_{12}}{p_{11}} &= 1 \\
(1-\pi_1) \frac{p_{11}}{p_{12}} + 2(1-\pi_1) &= 2
\end{aligned}$$

which are linearly dependent, solving for the ratio

$$\frac{p_{12}}{p_{11}} = \frac{1}{2}\frac{1-\pi_1}{\pi_1}$$

Notice the goods in the second state are cheaper, as it is a more "abundant state". So, equilibrium allocations are

$$\begin{aligned}
x_{111} &= x_{211} = \pi_1 \\
x_{112} &= x_{212} = 1-\pi_1 \\
x_{121} &= x_{221} = 2\pi_1 \\
x_{221} &= x_{222} = 2-2\pi_1
\end{aligned}$$

2. *Compute Radner Equilibrium.*

Solution 7 *In this case you can pick the spot prices as the Arrow-Debreu equilibrium prices p. Based on the above calculation you can "guess" that*

$$\begin{aligned}
q_1 m_{11} &= 2\pi_1 p_{11} - 2p_{11} = -2p_{11}(1-\pi_1) = -q_2 m_{21} \\
q_1 m_{12} &= 2p_{11}(1-\pi_1) = -q_2 m_{22}
\end{aligned}$$

Checking feasibility is trivial, optimality then follows and so is market clearing.

3. *Show that the final allocation of goods is the same in both cases.*

Solution 8 *Obvious from the above construction.*