Econ 7050
HW#3
Suggested Solutions
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Problem 1: *Prove that for any two distributions F,G such that G is a mean-preserving spread of F, any risk averse individual (whose preferences over lotteries are represented by vNM utility) prefers F to G.*

This can be shown using the definition and example of mean-preserving spread provided in MWG pg 197. We have a lottery over x with a distribution $F(\cdot)$, and we then randomize again so that payoffs are $x+z$, where the mean of $x+z$ is x. Here we say z is distributed $H_x(\cdot)$. The payoffs in the reduction of this compound lottery are distributed $G(\cdot)$. By definition of mean preserving spread, we write:

$U(G) = \int u(x)dG(x) = \int \left(\int u(x+z)dH_x(z)\right) dF(x)$

And by concavity of $u(x)$ (risk averse individual):

$\int \left(\int u(x+z)dH_x(z)\right) dF(x) \leq \int u\left(\int (x+z)dH_x(z)\right) dF(x) = \int u(x)dF(x) = U(F)$

Thus we have

$U(G) \leq U(F)$

So F is preferred to G for any risk averse individual.

Problem 2: *Prove that for any two distributions F, G, such that*

$\int_{\underline{z}}^{z} G(t)dt \geq \int_{\underline{z}}^{z} F(t)dt, \;\; z \geq \underline{z}$

any risk averse individual (whose preferences over lotteries are represented by vNM utility) prefers F to G

We need to show that $U(F) - U(G) \geq 0$.

$U(F) - U(G) = \int_{\underline{z}}^{\bar{z}} u(x)[f(x) - g(x)]dx$

We can use integration by parts to get:

$[u(x)(F(x) - G(x))]_{\underline{z}}^{\bar{z}} - \int_{\underline{z}}^{\bar{z}} u'(x)(F(x) - G(x))dx$

$= \int_{\underline{z}}^{\bar{z}} u'(x)(F(x) - G(x))dx$

$= \int_{\underline{z}}^{\bar{z}} u'(x)(G(x) - F(x))dx$

And we use integration by parts again to get

$= [u'(x) \int_{\underline{z}}^{x}(G(t) - F(t))dt]_{\underline{z}}^{\bar{z}} - \int_{\underline{z}}^{\bar{z}} u''(x)[\int_{\underline{z}}^{x}(G(t) - F(t))dt]dx$

We know that:

$u'(x) > 0$, $u''(x) < 0$

$\int_{\underline{z}}^{x}(G(t) - F(t))dt \geq 0$

So we can say that

$U(F) - U(G) \geq 0$

Which means that F is preferred to G.

Problem 3: *Show that preferences of an individual with decreasing relative risk aversion exhibit decreasing absolute risk aversion, but the converse is not necessarily true.*

We begin with the definitions:

The coefficient of relative risk aversion: $\sigma(x) = -\frac{u''(x)}{u'(x)}x$

The coefficient of absolute risk aversion: $r(x) = -\frac{u''(x)}{u'(x)}$

We first want to show that $\sigma'(x) < 0 \Rightarrow r'(x) < 0$

$\sigma(x) = r(x)x$

$\sigma'(x) = r'(x)x + r(x)$

So if $\sigma'(x) < 0$, then $r'(x)x + r(x) < 0$. We know that $x > 0$, and $r(x) > 0$, so it must be the case that $r'(x) < 0$. Thus, preferences with decreasing relative risk aversion exhibit decreasing absolute risk aversion.

We now want to show that $r'(x) < 0 \nRightarrow \sigma'(x) < 0$

$r(x) = \frac{\sigma(x)}{x}$

$r'(x) = \frac{\sigma'(x)x - \sigma(x)}{x^2}$

So if $r'(x) < 0$, then $\frac{\sigma'(x)x - \sigma(x)}{x^2} < 0$

This implies $\sigma'(x)x - \sigma(x) < 0$ or $\sigma'(x) < r(x)$

We know that for concave, non-decreasing utility, $r(x) > 0$, so it is possible that when $r'(x) < 0$, $\sigma'(x) \geq 0$. Thus, preferences with decreasing absolute risk aversion do not necessarily exhibit decreasing relative risk aversion.

Problem 4: *True or False: if preferences of an individual exhibit decreasing relative risk aversion, his risk premium decreases with wealth. Prove your answer.*

True (or False if you decide that the wording implies strictly decreases with wealth). We know from the previous problem that if preferences exhibit decreasing relative risk aversion, then they exhibit decreasing absolute risk aversion. By definition of decreasing absolute risk aversion:

$E(u(w + x) > u(w + z) \Rightarrow E(u(w' + x) > u(w' + z), \;\; w' > w$

$u(c(w + x)) > u(w + z) \Rightarrow u(c(w' + x)) > u(w' + z)$

$c(w + x) > w + z \Rightarrow c(w' + x) > w' + z$

$c(w + x) > w \Rightarrow c(w' + x) > w'$

Which implies that

$w - c(w + x) \geq w' - c(w' + x)$

Add the expected value of x to both sides:

$E(w + x) - c(w + x) \geq E(w' + x) - c(w' + x)$

Or:

$\pi(F(w + x)) \geq \pi(F(w' + x))$

Recall that $w' > w$, so the risk premium, π, decreases (or is at least non-increasing) with wealth.