HW3
Econ 4211

October 3, 2008

1. Two villagers want to hire several Samurai days to protect their crops. It costs 4 bags of rice to hire a Samurai for one day. Assume (for simplicity) that the villagers' preferences are defined over bundles of rice, (consumed privately) and "safety" (number of Samurai days in the village). The utility of the two villagers, a, b are:

$$\begin{aligned} u^a(r_a, g) &= 4\ln g + \ln r_a \\ u^b(r_b, g) &= 16\ln g + \ln r_b \end{aligned}$$

where r_a is the rice (in bags) consumed by villager a and r_b is the rice consumed by villager b, g is the number of Samurai days hired in the village. The first villager holds initially $R_a = 100$ bags of rice and the second one has $R_b = 120$.

   (a) Write down the resource constraint of the village (in terms of bags of rice).

   Solution 1 $r_a + r_b + 4g = 220$

   (b) Formulate the problem of the benevolent planner (the wise man) for this village.

   Solution 2

$$\max_{r_b, r_a, g} u^a(r_a, g) + u^b(r_b, g)$$
$$s.t.$$
$$r_a + r_b + 4g = 220$$

(c) Solve the problem. What is the optimal number of Samurai that the villagers should hire together? (Hint: in this case the wise man has to solve for both the rice consumption and the number of samurai – use all the optimality conditions from the wise-man problem.)

Solution 3 *Form the lagrangian:*

$$L(r_b, r_a, g, \lambda) = u^a(r_a, g) + u^b(r_b, g) - \lambda(r_a + r_b + 4g - 220)$$

and write the first order conditions:

$$\begin{aligned} u_r^a(r_a, g) - \lambda &= 0 \\ u_r^b(r_b, g) - \lambda &= 0 \\ u_g^a(r_a, g) + u_g^b(r_b, g) - 4\lambda &= 0 \\ r_a + r_b + 4g - 220 &= 0 \end{aligned}$$

where $u_r^a(r_a, g)$ is the marginal utility for good r of villager a, $u_g^b(r_b, g)$ is the marginal utility for good g of villager b, etc.
Eliminate λ from the first order conditions:

$$\begin{aligned} \frac{u_g^a(r_a, g)}{u_r^a(r_a, g)} + \frac{u_g^b(r_b, g)}{u_r^b(r_b, g)} &= 4 \qquad \textit{(Samuelson's cond)} \\ u_r^a(r_a, g) &= u_r^b(r_b, g) \\ r_a + r_b + 4g - 220 &= 0 \end{aligned}$$

and solve:

$$\begin{aligned} \frac{4r_a + 16r_b}{g} &= 4 \\ r_a &= r_b \\ r_a + r_b + 4g - 220 &= 0 \end{aligned}$$

so

$$\begin{aligned} 20r_a &= 4g \\ 22r_a &= 220 \end{aligned}$$

and therefore $r_a = 10 = r_b$, $g = 50$.

(d) Assume now the villagers decide how much protection to hire without consulting each other or the wise man (each decides on own expense on the Samurai taking the decision of the other as fixed.) How many Samurai will be hired then?

Solution 4 *The sum of the contributions, $g_a + g_b$ is the total amount of samurai in the village, g.*
Villager a solves:

$$\max_{r_a, g_a} u^a(r_a, g)$$
$$s.t.$$
$$r_a + 4g_a = 100$$

Optimality conditions:

$$\frac{u_g^a(r_a, g)}{u_r^a(r_a, g)} = 4$$
$$r_a + 4g_a = 100$$

the first equation implies $\frac{4r_a}{g_a+g_b} = 4$*, then* $g_a + g_b + 4g_a = 100$*, Solution is:* $g_a = -\frac{1}{5}g_b + 20$*. This is the "reaction" of villager a to decisions (how many samurai to hire,* g_a*) of villager* g_b*.*
Villager b.

$$\max_{r_a, g_b} u^b(r_b, g)$$
$$s.t.$$
$$r_b + 4g_b = 120$$

Optimality conditions (follow from the first order conditions after eliminating λ, the Lagrange multiplier):

$$\frac{u_g^b(r_b, g)}{u_r^b(r_b, g)} = 4$$
$$r_b + 4g_b = 120$$

the first equation implies $\frac{16r_a}{g_a+g_b} = 4$, *then* $\frac{(g_a+g_b)}{4} + 4g_b = 120$, *Solution is:* $g_b = -\frac{1}{17}g_a + \frac{480}{17}$.
The equilibrium contributions have to satisfy then

$$\begin{aligned} g_a &= -\frac{1}{5}g_b + 20 \\ g_b &= -\frac{1}{17}g_a + \frac{480}{17} \end{aligned}$$

Solution is: $g_a = \frac{305}{21}, g_b = \frac{575}{21}$. *The amount of protection hired in this case is* $g_a + g_b = \frac{305}{21} + \frac{575}{21} = \frac{880}{21} = 41.905$

(e) Do the two answers (c,d) differ?

Solution 5 *Yes,* $41.905 < 50$, *the optimal amount.*

Why?

Solution 6 *When deciding on the amount of protection to hire each villager overlooks the positive externalities imposed on the neighbour by his decision.*

What problem does the Wise Man help the villagers to solve?

Solution 7 *Free-rider problem.*

Problem 8 ***Problem 9*** *Andrew, Beth and Cathy live in Lindville. Bike paths (measured in miles, Q) are commonly used (public good). Andrew's willingness to pay (inverse demand) is* $P_a(Q) = 6 - Q/2$, *Beth's one is* $P_b(Q) = 18 - Q$ *and Cathy's is* $P_c(Q) = 24 - 3Q$. $MC(Q) = 21$.

1. *Assume the government can perfectly observe the willingness to pay for each. How long should be the bike path? How much should each contribute?*

   Solution 10 *Total willingness marginal willingness to pay should equal marginal cost by the Samuelson's condition:*

$$\begin{aligned} P_a(Q) + P_b(Q) + P_c(Q) &= 21 \\ 6 - Q/2 + 18 - Q + 24 - 3Q &= 21 \end{aligned}$$

   Solution is: $Q = 6$. *Optimal contributions per mile:* $P_a(6) = 6 - 6/2 = 3, P_b(6) = 18 - 6 = 12$, $P_c(6) = 24 - 3 * 6 = 6$.

2. *Is equal tax per mile on all will lead to an optimal provision of bike paths?*

   Solution 11 *No. If the tax per mile is 7, then Andrew is not ready to contribute (his willingness to pay for the first mile is 6), so* $Q = 0$.

Problem 12 *Consider the confectioner and the doctor example discussed in class (the value for doctor to operate is 60, the value for the confectioner is 40). Assume in addition doctor can install a soundproofing device that costs 18, in which case he can operate in the same building as the confectioner. This is known to the confectioner.*

1. *What is the socially optimal outcome (arrangement) in this case?*

   ANSWER: The total surplus is maximized when both operate and the sound-proofing device is installed, so that the surplus is $60 + 40 - 18 = 82$.

2. *Describe the outcomes (who operates and what payments are made) in all the cases below and compare them to the optimal one.*

   (a) *Doctor has a right to operate in a quiet environment, negotiation is possible (costs of negotiation are negligible)*

   Answer: The confectioner offers to pay to the doctor P, *sufficient amount to install the sound-proofing device, but not more than the value of the confectioner's business. The device is installed, both operate, total surplus is* $(60 + P - 18) + (40 - P) = 82$, *so the outcome is optimal.*

   (b) *Confectioner has a right to make noise, negotiation is possible.*

   Answer: Doctor imposes the device and pays for it, both operate, total surplus is $(60 - 18) + 40 = 82$, *so the outcome is optimal.*

   (c) *Negotiation is impossible (too expensive), government is unaware of the soundproofing technology available to the doctor and imposes a Pigouvian tax on the confectioner.*

   Answer. Instead of paying the tax equal to the damage created in the eyes of the government, 60, the confectioner shuts down, only doctor operates, the outcome is suboptimal, as the total surplus is $60 < 82$.

(d) *Negotiation is impossible (too expensive), government is unaware of the soundproofing technology available to the doctor, a permit to make noise is offered for sale. (Only the holder of the permit has a right to make the noise.)*

Answer: The confectioner is ready to pay up to 40 for the permit. Doctor has to options: either to buy the permit, or, if the permit is sold to the confectioner, install the device.

The doctor would have to pay at least 40 for the permit, because as long as the price for the permit is below 40 the confectioner would try to "outbid" the doctor. So, if the price at which the permit is sold is above 18 but below 40, the doctor will install the device and pay for it. As long as the price of the permit is below 40, the confectioner is better off buying the permit. The price can not be below 18 because otherwise the doctor would want to buy it and the confectioner will offer a higher price than 18 to get the permit.

The allocation is efficient, as the device is installed and both operate.