A Primer on Difference Equations

1. First-Order Difference Equations

Consider the following differential equation:

$$y_{t+1} + \beta y_t = \kappa, \tag{1.1}$$

where y is a variable of interest, t is time, and β and κ are constants. The solution to this differential equation is the sum of a particular integral y^p and a complementary function y^c:

$$y_t = y^p + y^c. \tag{1.2}$$

The solution depends on the value of β.

The Particular Integral

The particular integral is any solution to the full nonhomogenous equation. For example, assume that y is constant over time. Then, $y_{t+1} = y_t$ and

$$y^p = \kappa/(1+\beta) \qquad (\beta \neq -1). \tag{1.3}$$

The Complementary Function

The complementary function is the solution to the homogenous equation

$$y_{t+1} + \beta y_t = 0, \tag{1.4}$$

We guess that the solution is of the form

$$y^c = \Gamma \rho^t, \tag{1.5}$$

where Γ and ρ are an undetermined coefficients. The value of ρ is found as follows. Note that if our guess holds, $y_{t+1} = \Gamma\rho^{(}t+1)$, and we can write (1.4) as:

$$\Gamma\rho^{t+1} + \beta\Gamma\rho^t = 0, \tag{1.6}$$

which implies that $\rho = -\beta$.

The Full Solution

To obtain the full solution, we must identify Γ. As long as $\beta \neq 0$, we have:

$$\begin{aligned} y_t &= y^p + y^c, \\ &= \frac{\kappa}{(1+\beta)} + \Gamma\left(-\beta\right)^t. \end{aligned} \tag{1.7}$$

We find Γ by imposing an initial condition. In this case, we assume that $y_0 = \bar{y}$ at time $t = 0$. Also, at time $t = 0$, we have $y_0 = \frac{\kappa}{(1+\beta)} + \Gamma(-\beta)^0$, such that $\Gamma = \bar{y} - \kappa/(1+\beta)$. The full solution is then

$$y_t = \frac{\kappa}{(1+\beta)} + \left(\bar{y} - \frac{\kappa}{(1+\beta)}\right)\left(-\beta\right)^t \qquad (\beta \neq -1) \tag{1.8}$$

In the case where $\beta = -1$, the differential equation reduces to

$$y_{t+1} - y_t = \kappa. \tag{1.9}$$

In this case, the solution is found by straight integration to be

$$y_t = \kappa t + \Gamma, \tag{1.10}$$

where Γ is an undetermined coefficient. Using our initial condition, $y_0 = \bar{y} = \kappa 0 + \Gamma$, such that $\Gamma = \bar{y}$. The full solution is

$$y_t = \bar{y} + \kappa t \qquad (\beta = -1). \tag{1.11}$$

So, the solutions are:

$$y_t = \begin{cases} \kappa/(1+\beta) + \left[\bar{y} - \kappa/(1+\beta)\right]\left(-\beta\right)^t & \text{if } \beta \neq -1 \\ \bar{y} + \kappa t & \text{if } \beta = -1 \end{cases}. \tag{1.12}$$

2. Second-Order Differential Equations

The differential equation can be written as:

$$y_{t+2} + \alpha y_{t+1} + \beta y_t = \kappa, \tag{2.1}$$

where y is a variable, t is time, and α, β, and κ are constants.

The Particular Integral

The particular integral is any solution to the full nonhomogenous equation. For example, assume that y is constant over time. Then, $y_{t+2} = y_{t+1} = y_t$ and

$$y^p = \kappa/(1+\alpha+\beta) \qquad (\alpha+\beta \neq -1). \tag{2.2}$$

If $\alpha+\beta = -1$, then assume that $y_t = \eta t$. This implies that $y_{t+2} = \eta(t+2)$, $y_{t+1} = \eta(t+1)$, and (at $t = 0$)$(2+\alpha)\eta = \kappa$. Thus, $\eta = \kappa/(\alpha+2)$, and the particular integral is

$$y^p = \frac{\kappa}{(\alpha+2)}t \quad (\alpha+\beta = -1, \alpha \neq -2). \tag{2.3}$$

Finally, if $\alpha+\beta = -1$ and $\alpha = -2$, we try $y_t = \eta t^2$. It implies that $y_{t+2} = \eta(t+2)^2$, $y_{t+1} = \eta(t+1)$, and (at $t = 0$) $(4+\alpha)\eta = \kappa$. Then, $\eta = \kappa/2$ and

$$y^p = \frac{\kappa}{2}t^2 \quad (\alpha+\beta = -1, \alpha = -2). \tag{2.4}$$

So, the particular integral is:

$$y^p = \begin{cases} \kappa/(1+\alpha+\beta) & \text{if } \alpha+\beta \neq -1 \\ [\kappa/(\alpha+2)]\,t & \text{if } \alpha+\beta = -1 \text{ and } \alpha \neq -2\ . \\ [\kappa/2]\,t^2 & \text{if } \alpha+\beta = -1 \text{ and } \alpha = -2 \end{cases} \tag{2.5}$$

The Complementary Function

The complementary function is the solution to the homogenous equation

$$y_{t+2} + \alpha y_{t+1} + \beta y_t = 0. \tag{2.6}$$

We guess that the solution is of the form

$$y^c = \Gamma \rho^t. \tag{2.7}$$

This solution implies that $y_{t+2} = \Gamma\rho^{t+2}$ and $y_{t+1} = \Gamma\rho^{t+1}$. Substituting in (2.5), we find

$$\Gamma\rho^{t+2} + \alpha\Gamma\rho^{t+1} + \beta\Gamma\rho^t = 0. \tag{2.8}$$

The solution to this requires to solve the quadratic form:

$$\rho^2 + \alpha\rho + \beta = 0. \tag{2.9}$$

The solution to the quadratic form is:

$$\rho_1, \rho_2 = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}. \tag{2.10}$$

This implies that

$$\rho_1 + \rho_2 = -\alpha$$

$$\rho_1 \rho_2 = \beta$$

There are three possible cases.

Case 1: Distinct Real Roots $\alpha^2 > 4\beta$. In this case, the solution is

$$y^c = \Gamma_1 \rho_1^t + \Gamma_2 \rho_2^t. \tag{2.11}$$

where $\rho_1 + \rho_2 = -\alpha$ and $\rho_1 \rho_2 = \beta$.

Case 2: Repeated Real Roots $\alpha^2 = 4\beta$. In this case,

$$y^c = \Gamma_1 \rho^t + \Gamma_2 t \rho^t, \tag{2.12}$$

and $\rho = -\alpha/2$.

Case 3: Complex Roots $\alpha^2 < 4\beta$.

$$y^c = \Gamma_1 \rho_1^t + \Gamma_2 \rho_2^t, \tag{2.13}$$

where $\rho_1 = h + vi$ and $\rho_2 = h - vi$, for $h = -\alpha/2$, $v = \left(\sqrt{4\beta - \alpha^2}\right)/2$, and $i = \sqrt{-1}$. We can rewrite equation (2.13) as:

$$y^c = R^t \left(\Gamma_3 \cos \theta t + \Gamma_4 \sin \theta t\right), \tag{2.14}$$

where $\Gamma_3 = \Gamma_1 + \Gamma_2$, $\Gamma_4 = \left(\Gamma_1 - \Gamma_2\right) i$, $R = \sqrt{\beta^2}$, and θ defined such that $\cos \theta = h/R$ and $\sin \theta = v/R$.

The Full Solution

The full solution requires to identify Γ. Consider first the distinct real rood case ($\alpha^2 > 4\beta$). In this case $y = y^p + y^c$ and

$$y_t = y^p + \Gamma_1 \rho_1^t + \Gamma_2 \rho_2^t, \tag{2.15}$$

where $\rho_1 + \rho_2 = -\alpha$, $\rho_1 \rho_2 = \beta$, and y^p defined in equation (2.5). We require two conditions to find Γ_1 and Γ_2. For example, we impose $y_0 = \bar{y}$ and $y_1 = \hat{y}$. Then,

$$y_0 = \bar{y} = y^p + \Gamma_1 + \Gamma_2$$

$$y_1 = \hat{y} = y^p + \rho_1 \Gamma_1 + \rho_2 \Gamma_2.$$

These two equations (with two unknowns) can be solved for Γ_1 and Γ_2.

In the repeated real root case ($\alpha^2 = 4\beta$), we find

$$y_t = y^p + \Gamma_1 \rho^t + \Gamma_2 t \rho^t, \tag{2.16}$$

where $\rho = -\alpha/2$ and y^p defined in equation (2.5). We impose $y_0 = \bar{y}$ and $y_1 = \hat{y}$ to obtain:

$$y_0 = \bar{y} = y^p + \Gamma_1$$

$$y_1 = \hat{y} = y^p + \rho_1 \Gamma_1 + \rho_2 \Gamma_2.$$

These can be solved for Γ_1 and Γ_2.

Finally, in the complex root case, $\alpha^2 < 4\beta$, we find

$$y_t = y^p + R^t \left(\Gamma_3 \cos\theta t + \Gamma_4 \sin\theta t\right), \tag{2.17}$$

where $h = -\alpha/2$, $v = \left(\sqrt{4\beta - \alpha^2}\right)/2$, $R = \sqrt{\beta^2}$, and θ defined such that $\cos\theta = h/R$ and $\sin\theta = v/R$. We use $y_0 = \bar{y}$ and $y_1 = \hat{y}$ to obtain:

$$y_0 = \bar{y} = y^p + \Gamma_3,$$

$$y_1 = \hat{y} = y^p + R\Gamma_3 \cos\theta + R\Gamma_4 \sin\theta,$$

since $\cos 0 = 1$ and $\sin 0 = 0$. The two equations can be solved for Γ_3 and Γ_4.

References

Chiang, A.C. (1984) *Fundamental Methods of Mathematical Economics*, Third Edition, McGraw Hill.