# Elements of Linear Algebra: Q&A

A *matrix* is a rectangular array of objects (*elements* that are numbers, functions, etc.) with its size indicated by the number of rows and columns, i.e., an $m \times n$ matrix $A$ with $m$ rows and $n$ columns.

If $A$ is an $m \times n$ matrix, $A^T$ is an $n \times m$ matrix.

The *determinant* of a matrix is the absolute value of the sum of the diagonal elements. The determinant is only defined for a square matrix. The determinant of a matrix can be computed using the *Laplace expansion* where a row or column is expanded in terms of *minors* and *cofactors*.

An *orthogonal* matrix is an invertible $n \times n$ matrix $Q$ with the property $Q^{-1} = Q^T$.

# Elements of Linear Algebra: Q&A

Given a system of $m$ linear equations in $n$ variables $x_i$ ($i = 1, \ldots, n$), written as $A\mathbf{x} = \mathbf{b}$, the system is either

1. *Consistent*, with a unique (one) solution $\mathbf{x}$.
2. *Consistent*, with infinitely many possible solutions.
3. *Inconsistent* with no solutions.

If $n > m$, the system has more unknowns than equations it is *underdetermined*. If the system is consistent, some of the variables can be chosen arbitrarily and the remaining variables defined in terms of the arbitrary ones.

If $n < m$, the system has more equations than unknowns it is *overdetermined*.

# Elements of Linear Algebra: Q&A

**P3.16. Invertible matrix properties** Assume that $A$ is an $n \times n$ invertible matrix. Which statements are true?

a. The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b}$ in $\mathbf{R}^n$.

b. The rows (and columns) of $A$ are linearly independent.

c. $\det(A) = 0$.

d. $A$ can be reduced (by elementary operations) to the identity matrix.

e. The rank of $A$ is $n$.

f. The rows of $A$ span $\mathbf{R}^n$.

# Elements of Linear Algebra: Q&A

**P3.18. Linear Independence** Consider the equations of combustion in which a mixture of CO, $H_2$, and $CH_4$ are burned with $O_2$ to form CO, $CO_2$ and $H_2O$.

$$CO + \frac{1}{2}O_2 = CO_2.$$

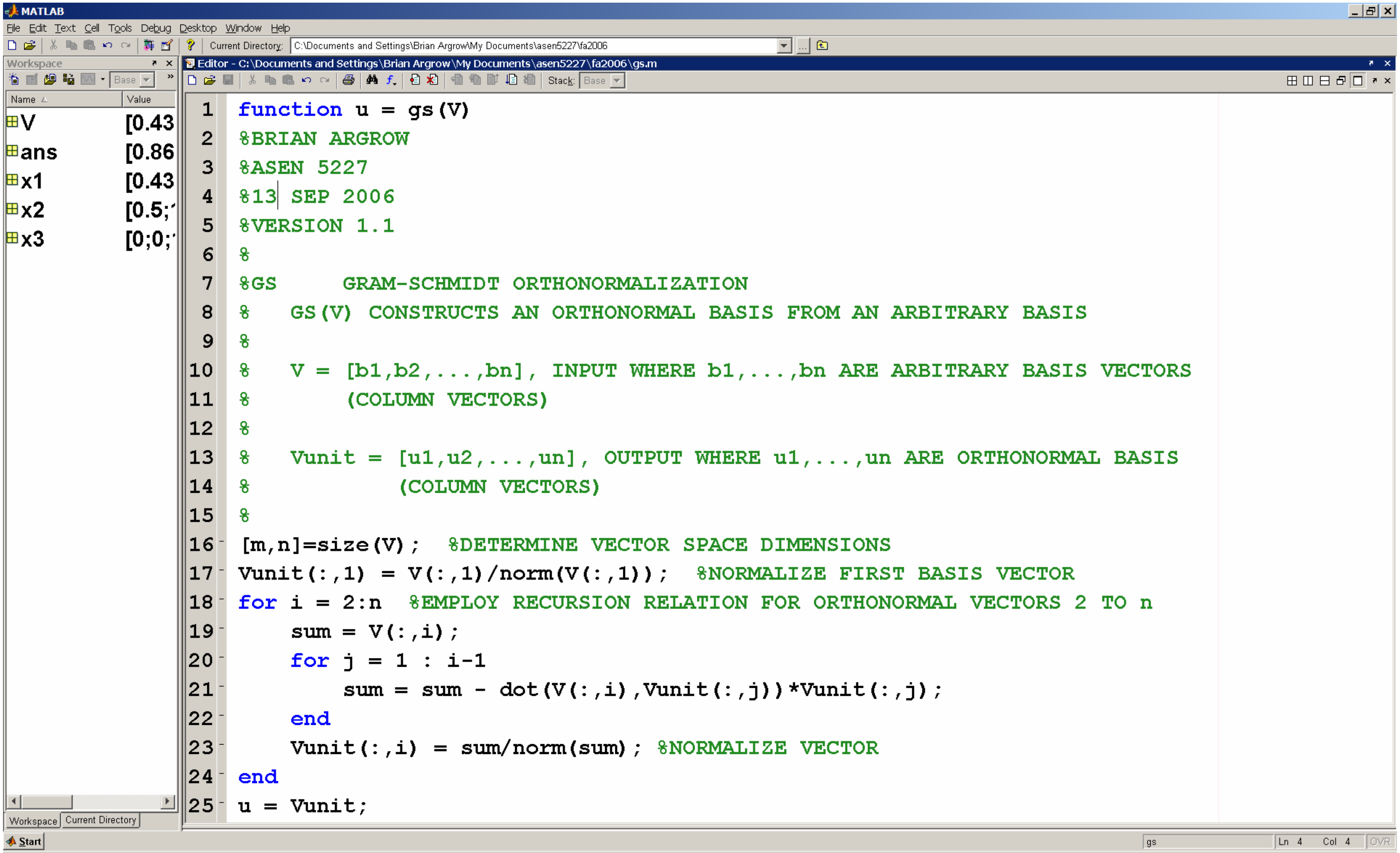
$$H_2 + \frac{1}{2}O_2 = H_2O.$$

$$CH_4 + 2O_2 = CO_2 + 2\,H_2O.$$

$$CH_4 + \frac{3}{2}O_2 = CO + 2\,H_2O.$$

Treating the compounds as real variables, determine if the equations are independent. If not, write the dependent equation(s) in terms of the independent ones.

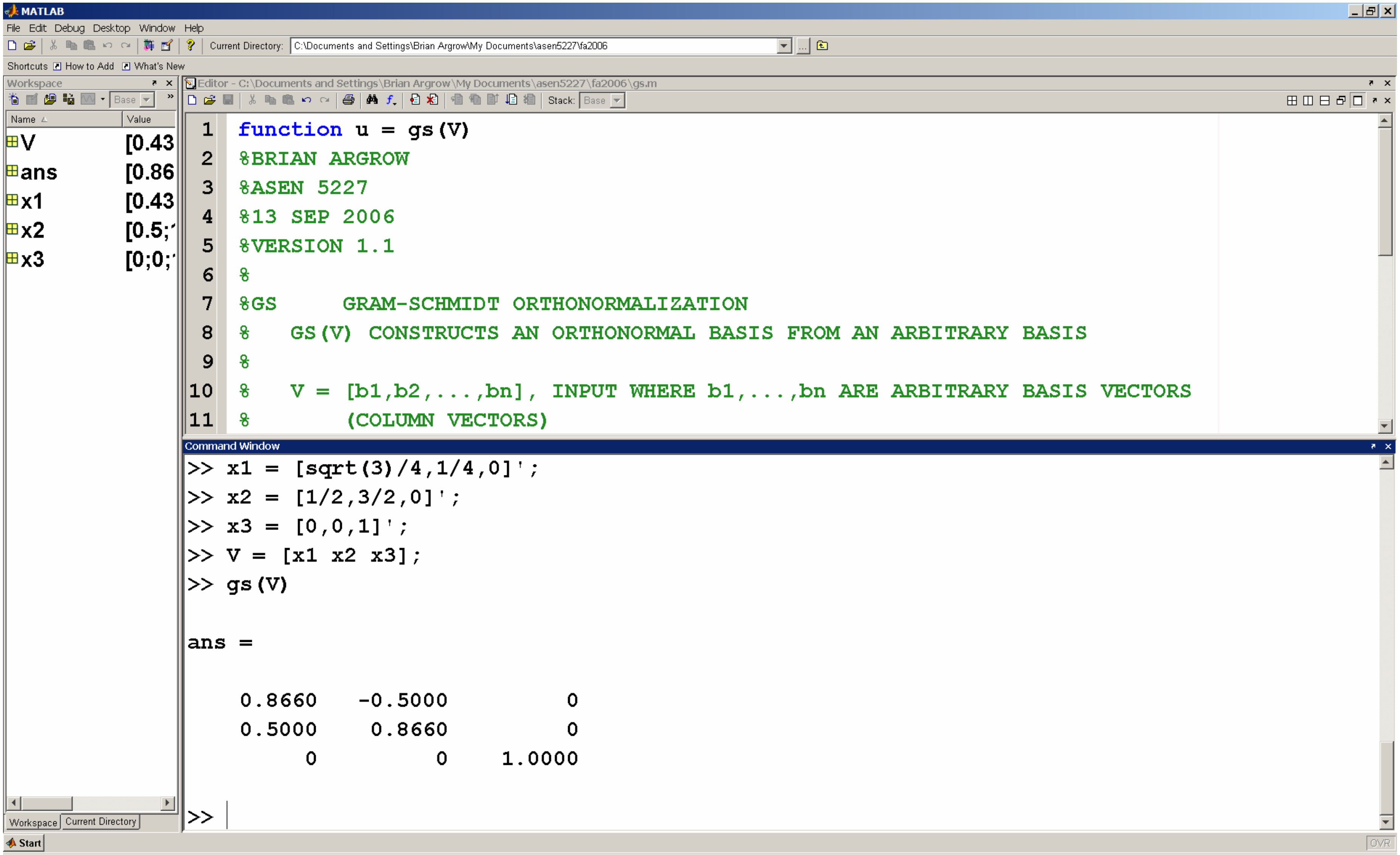
# Elements of Linear Algebra: AEM P2.35



| Name | Value |
|---|---|
| V | [0.43 |
| ans | [0.86 |
| x1 | [0.43 |
| x2 | [0.5; |
| x3 | [0;0; |

```
function u = gs(V)
%BRIAN ARGROW
%ASEN 5227
%13 SEP 2006
%VERSION 1.1
%
%GS      GRAM-SCHMIDT ORTHONORMALIZATION
%    GS(V) CONSTRUCTS AN ORTHONORMAL BASIS FROM AN ARBITRARY BASIS
%
%    V = [b1,b2,...,bn], INPUT WHERE b1,...,bn ARE ARBITRARY BASIS VECTORS
%        (COLUMN VECTORS)
%
%    Vunit = [u1,u2,...,un], OUTPUT WHERE u1,...,un ARE ORTHONORMAL BASIS
%            (COLUMN VECTORS)
%
[m,n]=size(V);  %DETERMINE VECTOR SPACE DIMENSIONS
Vunit(:,1) = V(:,1)/norm(V(:,1));  %NORMALIZE FIRST BASIS VECTOR
for i = 2:n  %EMPLOY RECURSION RELATION FOR ORTHONORMAL VECTORS 2 TO n
    sum = V(:,i);
    for j = 1 : i-1
        sum = sum - dot(V(:,i),Vunit(:,j))*Vunit(:,j);
    end
    Vunit(:,i) = sum/norm(sum); %NORMALIZE VECTOR
end
u = Vunit;
```

# Elements of Linear Algebra: AEM P2.35



| Name | Value |
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%
%    V = [b1,b2,...,bn], INPUT WHERE b1,...,bn ARE ARBITRARY BASIS VECTORS
%        (COLUMN VECTORS)
```

```
>> x1 = [sqrt(3)/4,1/4,0]';
>> x2 = [1/2,3/2,0]';
>> x3 = [0,0,1]';
>> V = [x1 x2 x3];
>> gs(V)

ans =

    0.8660   -0.5000         0
    0.5000    0.8660         0
         0         0    1.0000

>> 
```

# Elements of Linear Algebra

## Eigenvalues & Eigenvectors

As an engineer, you have undoubtedly been introduced to eigenvalues and possibly eigenvectors. We develop background here and will later make use of eigenvalues/vectors in the discussion of second and higher-order tensors.

Given the linear equation $A\mathbf{x} = \mathbf{b}$ the vector **x** is called the *eigenvector* (characteristic vector) and the scalar $\lambda$ is the *eigenvalue* (characteristic value) of matrix *A* that characterizes the length (and sense) of the eigenvector **x**.

The *spectrum* of *A* is the set of eigenvalues of *A* and the *spectral radius* of *A* is the absolute value of the largest eigenvalue.

# Elements of Linear Algebra

Example: Find eigenvalues and eigenvectors of

$$\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$$

Solution:

1. Compute roots of the characteristic polynomial

$$D(\lambda) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 5 & 4-\lambda & 0 \\ 3 & 6 & 1-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda)(1-\lambda) = 0$$

roots: $\lambda_1 = 3,\ \lambda_2 = 4,\ \lambda_3 = 1.$

# Elements of Linear Algebra

These roots are the eigenvalues. They form the spectrum with a spectral radius of 4.

2. Compute the eigenvectors:

$\underline{\lambda_1 = 3}$

$$\left.\begin{aligned} 0 &= 0 \\ 5x_1 + x_2 &= 0 \\ 3x_1 + 6x_2 - 2x_3 &= 0 \end{aligned}\right\} \rightarrow \text{set } x_1 = 1 \rightarrow \mathbf{x} = \begin{bmatrix} -1 \\ -5 \\ -27/2 \end{bmatrix} \text{ or } \begin{bmatrix} -2 \\ -10 \\ -27 \end{bmatrix}$$

$\underline{\lambda_2 = 4}$

$$\left.\begin{aligned} -x_1 &= 0 \\ 5x_1 &= 0 \\ 3x_1 + 6x_2 - 3x_3 &= 0 \end{aligned}\right\} \rightarrow \text{set } x_2 = 1 \rightarrow \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

# Elements of Linear Algebra

$\underline{\lambda_3 = 1}$

$$\left.\begin{aligned} 2x_1 &= 0 \\ 5x_1 + 3x_2 &= 0 \\ 3x_1 + 6x_2 &= 0 \end{aligned}\right\} \rightarrow \text{set } x_3 = 1 \rightarrow \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Properties of eigenvalues and eigenvectors of an *n*×*n* square matrix *A*:

1. *A* has at least one eigenvalue and at most *n* numerically different eigenvalues, but it may have fewer than *n*.
2. If **x** is an eigenvector of a matrix *A* corresponding to an eigenvalue $\lambda$, so is $k\mathbf{x}$, for any $k \neq 0$, i.e.,

   $A\mathbf{x} = \lambda\mathbf{x}$ implies $k(A\mathbf{x}) = A(k\mathbf{x}) = \lambda(k\mathbf{x})$.

# Elements of Linear Algebra

3. $M_\lambda$ is the *algebraic multiplicity*, the number of times the root $\lambda$ of the characteristic polynomial is repeated, and $m_\lambda$ is the *geometric multiplicity*, the number of independent eigenvectors corresponding to $\lambda$. According to property 1 above, the sum of algebraic multiplicities equals $n$ and in general $m_\lambda \le M_\lambda$.
4. A real matrix may have complex eigenvalues that occur in conjugate pairs and complex eigenvectors.
5. The eigenvalues of a symmetric matrix ($A^T = A$) are real.
6. The eigenvalues of a skew symmetric matrix ($A^T = -A$) are pure imaginary or zero.

# Elements of Linear Algebra

## Eigenvectors & Diagonalization

*Similar matrices* have the same spectrum (i.e., same eigenvalues),

$\hat{A} = T^{-1}AT$ : $n \times n$ matrix $\hat{A}$, is *similar* to $A$ for some $n \times n$ matrix $T$.

This is an important property, particularly for numerical analysis, to diagonalize (or nearly diagonalize) matrices for computing approximations to eigenvalues and eigenvectors.

The eigenvectors corresponding to a set of *distinct* eigenvalues form a linearly independent set. Thus, these eigenvectors form a basis.

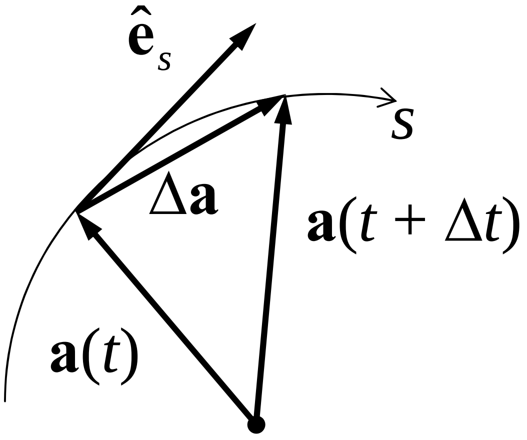
If an $n \times n$ matrix $A$ has a basis of eigenvectors, then $D = X^{-1}AX$ is diagonal with eigenvalues of $A$ as the entries on the main diagonal.

# Vector Calculus & General Coordinate Systems

The vector algebra included operations involving sums and products of vectors. The definitions and operations defined in the linear algebra provide the basis for linear transformations and matrix operations useful in tensor analysis.

The vector calculus allows us to apply the methods of differential and integral calculus in the general tensor analysis. We begin with the usual basic definitions and operations.

**Derivative of a Vector Function of a Scalar**



$$\frac{d\mathbf{a}}{dt} = \lim_{\Delta t\to 0}\frac{\mathbf{a}(t+\Delta t)-\mathbf{a}(t)}{\Delta t}$$

$$\Delta s = |\,\mathbf{a}\,|$$

$$\frac{d\mathbf{a}}{dt} = \lim_{\Delta t\to 0}\frac{\Delta\mathbf{a}}{\Delta s}\frac{\Delta s}{\Delta t} = \frac{ds}{dt}\hat{\mathbf{e}}_s$$

# Vector Calculus & General Coordinate Systems

## Product Rules

$$\frac{d}{dt}(\mathbf{a}\cdot\mathbf{b}) = \frac{d\mathbf{a}}{dt}\cdot\mathbf{b} + \mathbf{a}\cdot\frac{d\mathbf{b}}{dt}$$

$$\frac{d}{dt}(\mathbf{a}\times\mathbf{b}) = \frac{d\mathbf{a}}{dt}\times\mathbf{b} + \mathbf{a}\times\frac{d\mathbf{b}}{dt} \quad \text{(order preserved)}$$

Note that because a vector is composed of two distinct parts, magnitude and direction, a nonzero derivative could result from: a) a change in magnitude but not direction, b) a change in direction but not magnitude, or c) a change in both magnitude and direction as illustrated in the previous diagram.

# Vector Calculus & General Coordinate Systems

For case b), a constant length vector,

$$\frac{d}{dt}(\mathbf{a}\cdot\mathbf{a}=a^2)\rightarrow\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}=a\cdot\frac{da}{dt}=0\begin{cases}\dfrac{d\mathbf{a}}{dt}=\mathbf{0} & \begin{cases}|\mathbf{a}|=\text{const}\\ \dfrac{\mathbf{a}}{|\mathbf{a}|}=\text{const}\end{cases}\\ \mathbf{a}\perp\dfrac{d\mathbf{a}}{dt} & \begin{cases}|\mathbf{a}|=\text{const}\\ \dfrac{\mathbf{a}}{|\mathbf{a}|}\neq\text{const}\end{cases}\end{cases}$$

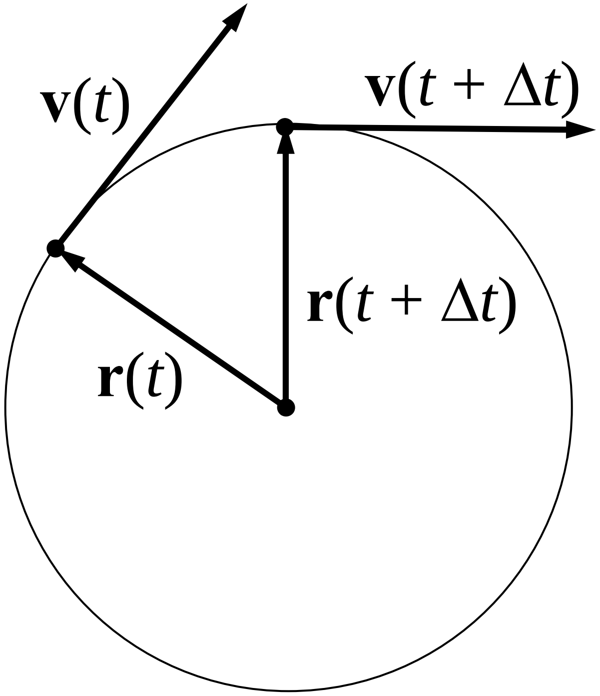
*In general coordinates, the base vectors are not necessarily constant in magnitude or direction,*

$$\mathbf{a}=a^i\mathbf{e}_i\rightarrow\frac{da^i}{dt}\mathbf{e}_i+a^i\frac{d\mathbf{e}_i}{dt}$$

By definition, *the base vectors of Cartesian systems have constant magnitude and direction* $\rightarrow d\mathbf{e}_i/dt=\mathbf{0}$.

# Vector Calculus & General Coordinate Systems

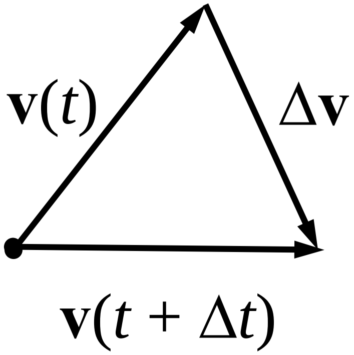
Example: Compute the acceleration of a body in a circular orbit.



$$\boldsymbol{\omega} = \omega\hat{\mathbf{e}}_z$$

$$\mathbf{r} = r\hat{\mathbf{e}}_r$$

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r} = v\hat{\mathbf{e}}_t$$



# Vector Calculus & General Coordinate Systems

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r}) = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt}$$

$$= \frac{v}{r}\frac{d\hat{\mathbf{e}}_z}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v} = \boldsymbol{\omega} \times \mathbf{v}$$

$$= \frac{v}{r}\hat{\mathbf{e}}_z \times (\boldsymbol{\omega} \times \mathbf{r}) = \frac{v^2}{r}[\hat{\mathbf{e}}_z \times (\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r)]$$

$$= \frac{v^2}{r}[(\hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_r)\hat{\mathbf{e}}_z - (\hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_z)\hat{\mathbf{e}}_r)]$$

$$= \frac{v^2}{r}\hat{\mathbf{e}}_r \quad \Leftarrow$$

# Vector Calculus & General Coordinate Systems

Example: Prove $\frac{d}{dt}\left[\mathbf{a}\cdot\left(\frac{d\mathbf{a}}{dt}\times\frac{d^2\mathbf{a}}{dt^2}\right)\right]=\mathbf{a}\cdot\left(\frac{d\mathbf{a}}{dt}\times\frac{d^3\mathbf{a}}{dt^3}\right)$

Solution:

$$\frac{d}{dt}\left[\mathbf{a}\cdot\left(\frac{d\mathbf{a}}{dt}\times\frac{d^2\mathbf{a}}{dt^2}\right)\right]=\frac{d\mathbf{a}}{dt}\cdot\cancelto{0}{\left(\frac{d\mathbf{a}}{dt}\times\frac{d^2\mathbf{a}}{dt^2}\right)}+\mathbf{a}\cdot\left(\cancelto{0}{\frac{d^2\mathbf{a}}{dt^2}\times\frac{d^2\mathbf{a}}{dt^2}}+\frac{d\mathbf{a}}{dt}\times\frac{d^3\mathbf{a}}{dt^3}\right)$$

$$=\mathbf{a}\cdot\left(\frac{d\mathbf{a}}{dt}\times\frac{d^3\mathbf{a}}{dt^3}\right)\quad\Leftarrow\ \text{Q.E.D.}$$

# Vector Calculus & General Coordinate Systems

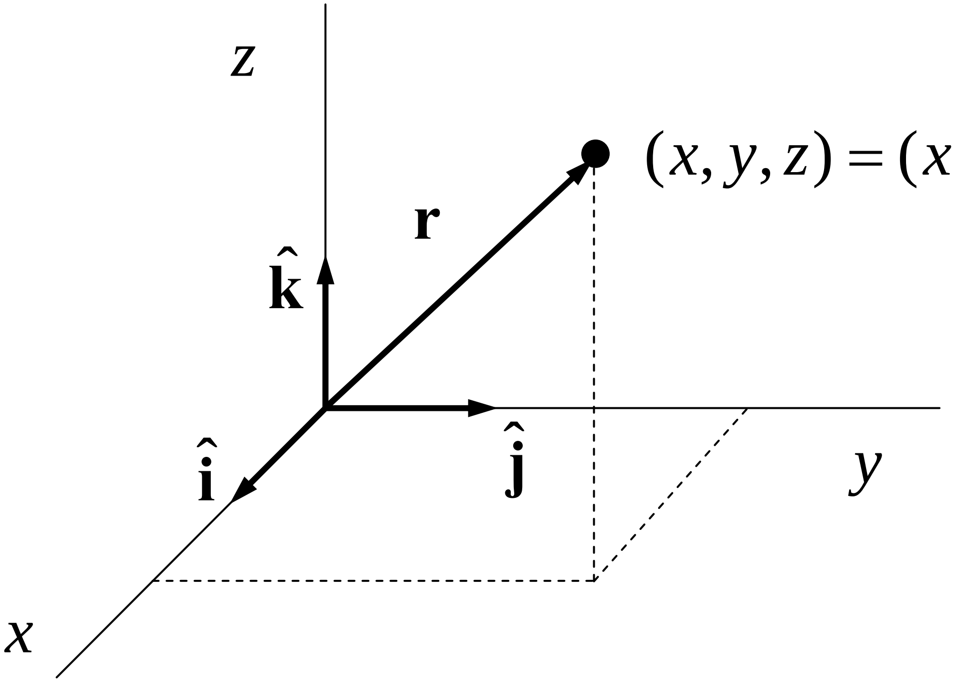
## Cartesian Coordinate Systems

A general Cartesian coordinate system is oblique, i.e., the basis vectors are generally not all mutually orthogonal. As stated earlier, however, the basis vectors of a Cartesian system are constant in magnitude and direction.

The usual convention is to refer to the familiar orthonormal Cartesian system as *the* Cartesian system, with basis vectors usually denoted as

$$\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}, \quad \{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\}, \quad \{\hat{\mathbf{i}}_i\}$$

# Vector Calculus & General Coordinate Systems



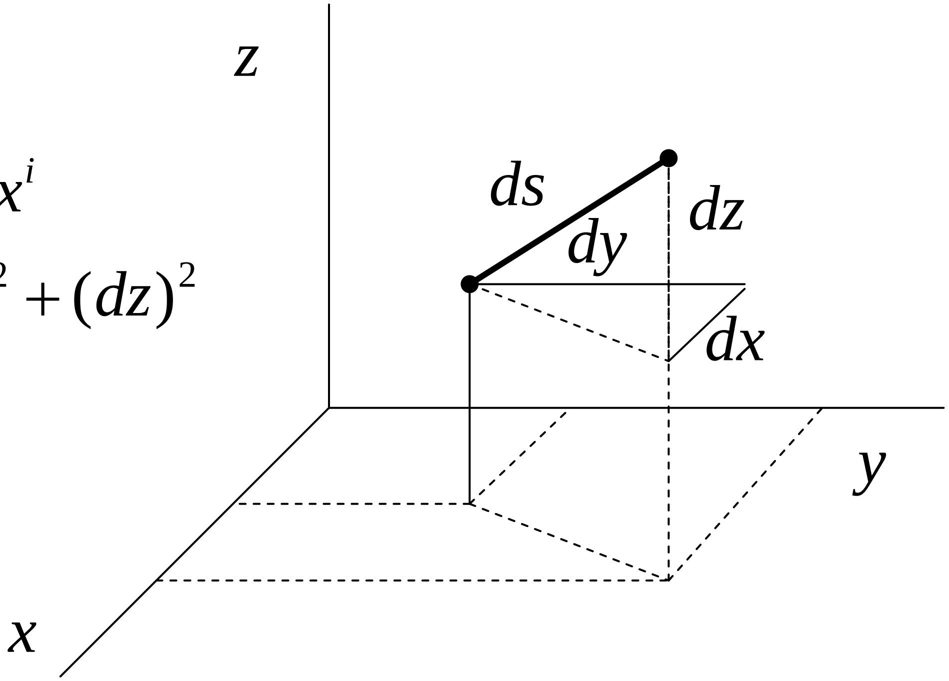
$$(x, y, z) = (x^1, x^2, x^3) = (x_1, x_2, x_3)$$

$$\mathbf{r} = x_j \hat{\mathbf{i}}_j$$

# Vector Calculus & General Coordinate Systems

In any coordinate system, the differential distance between two points is given by the differential arclength, computed from $d\mathbf{r}\cdot d\mathbf{r}$. In particular, for the Cartesian system,

$$
\begin{aligned}
d\mathbf{r}\cdot d\mathbf{r} &= (ds)^2 = dx^i dx^i \\
&= (dx)^2 + (dy)^2 + (dz)^2
\end{aligned}
$$



# Vector Calculus & General Coordinate Systems

## Curvilinear Coordinates

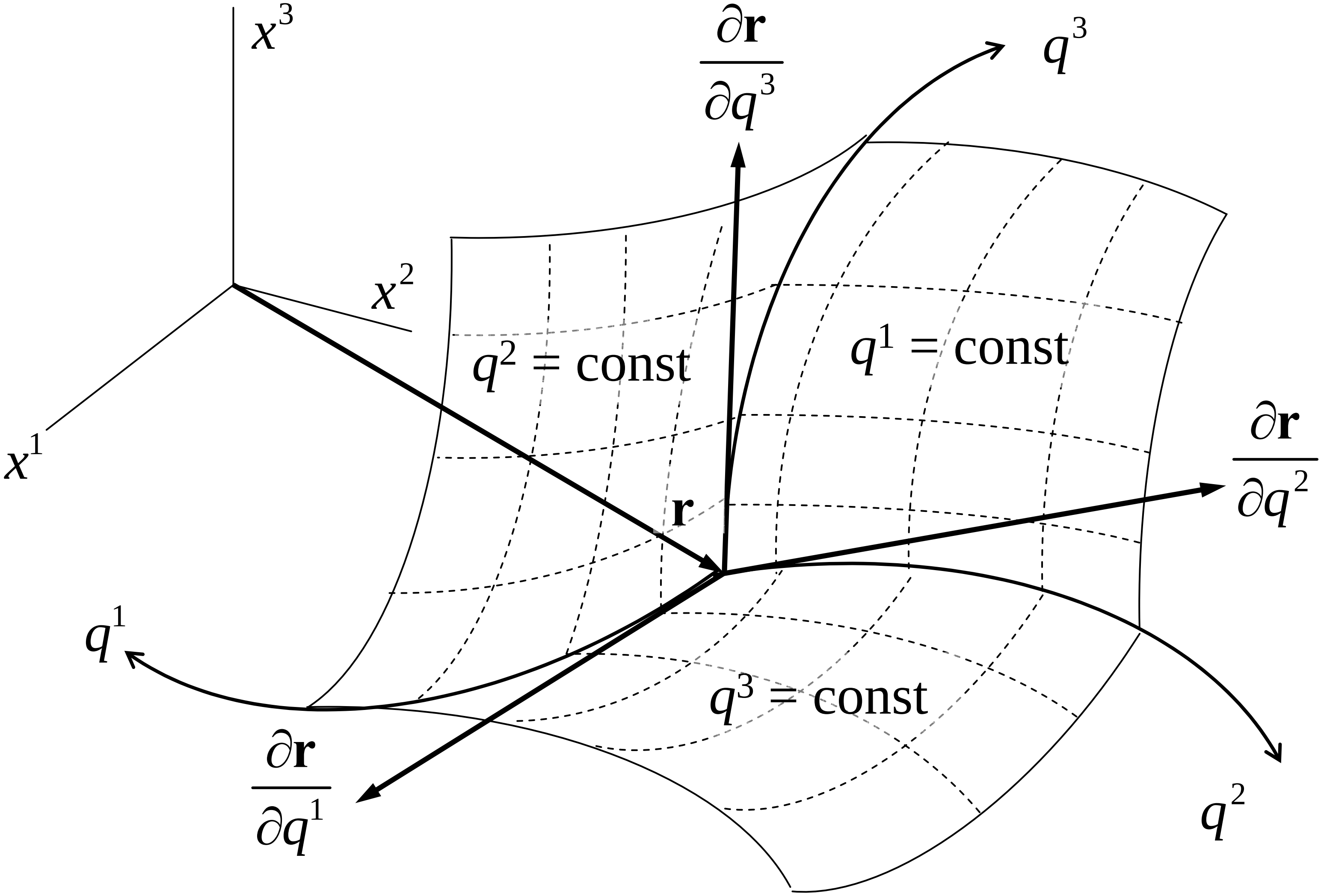
Define a coordinate system

$$(q^1, q^2, q^3)$$

with the coordinate transformation from the Cartesian system,

$$q^1 = q^1(x^1, x^2, x^3)$$
$$q^2 = q^2(x^1, x^2, x^3)$$
$$q^3 = q^3(x^1, x^2, x^3)$$

# Vector Calculus & General Coordinate Systems



# Vector Calculus & General Coordinate Systems

If the transformation is *linear*, it defines a *Cartesian* system. If the transformation is *nonlinear*, it defines a *curvilinear* system.

The *Jacobian* of the transformation is defined by the following determinant,

$$J = \left|\frac{\partial x^j}{\partial q^i}\right| = \begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{vmatrix} = \begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^2}{\partial q^1} & \frac{\partial x^3}{\partial q^1} \\ \frac{\partial x^1}{\partial q^2} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^3}{\partial q^2} \\ \frac{\partial x^1}{\partial q^3} & \frac{\partial x^2}{\partial q^3} & \frac{\partial x^3}{\partial q^3} \end{vmatrix}$$

# Vector Calculus & General Coordinate Systems

If $J \neq 0$, then $J^{-1}$ (inverse Jacobian) is defined and the inverse transformation is also defined,

$$x^1 = x^1(q^1, q^2, q^3)$$
$$x^2 = x^2(q^1, q^2, q^3)$$
$$x^3 = x^3(q^1, q^2, q^3)$$

The position “arrow” is

$$\mathbf{r} = \mathbf{r}(q^i), \quad i = 1, 2, 3$$

and a differential displacement is then

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 + \frac{\partial \mathbf{r}}{\partial q^3} dq^3 = \frac{\partial \mathbf{r}}{\partial q^i} dq^i$$

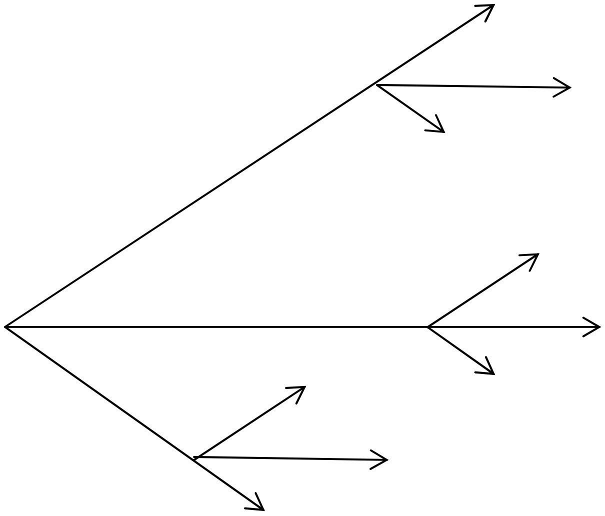
# Vector Calculus & General Coordinate Systems

The vectors $\partial \mathbf{r} / \partial q^i$ are tangent to the coordinate curves defined by the intersection of the coordinate surfaces ($q^i$ = const). Using these vectors, we define a *unitary basis*,

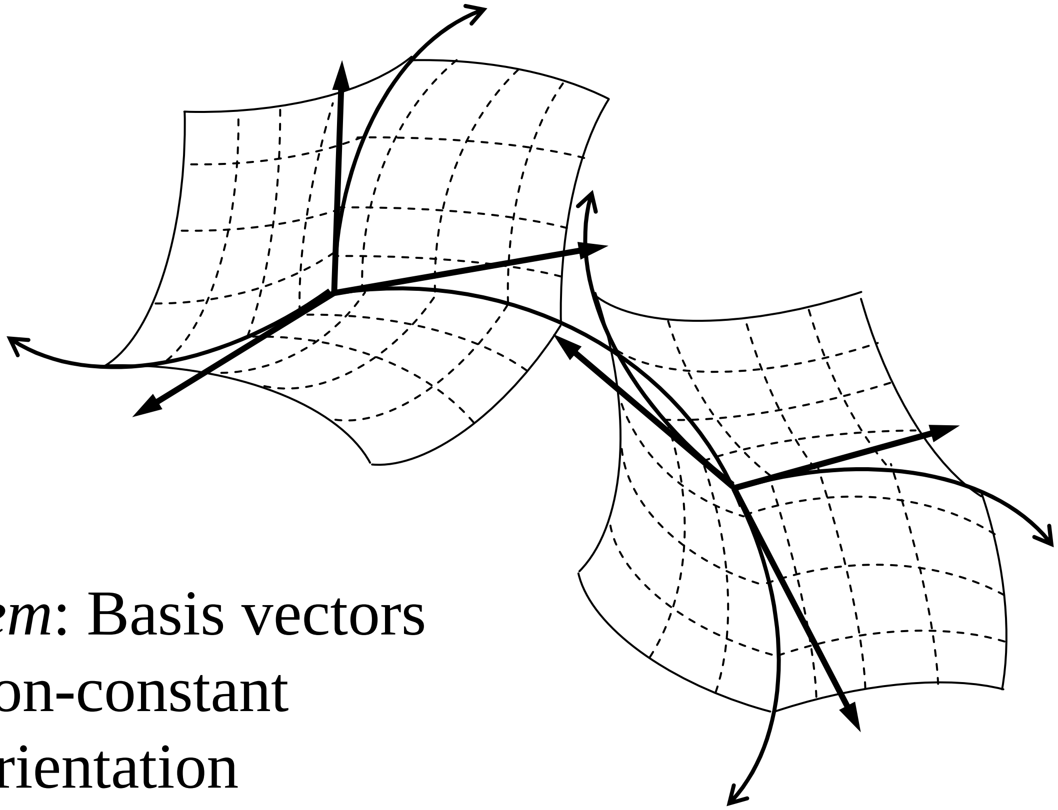
$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial q^i}, \quad i = 1, 2, 3$$

Note, in general, the orientation and magnitude of the basis vectors are not constant, e.g.,

# Vector Calculus & General Coordinate Systems



*Oblique-Cartesian system*: Basis vectors have constant magnitude and orientation



*Curvilinear system*: Basis vectors generally have non-constant magnitude and orientation

# Vector Calculus & General Coordinate Systems

The coordinate transformation was written for a general system in terms of the original Cartesian system. We almost always write the transformations in this manner. In terms of the original Cartesian system, the unitary basis is given by,

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial q^i} = \frac{\partial x^j}{\partial q^i}\hat{\mathbf{i}}_j, \quad i = 1,2,3$$

This is a linear system that is easily written in matrix format. The coefficient matrix is the *Jacobian matrix*,

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} \partial x^1/\partial q^1 & \partial x^2/\partial q^1 & \partial x^3/\partial q^1 \\ \partial x^1/\partial q^2 & \partial x^2/\partial q^2 & \partial x^3/\partial q^2 \\ \partial x^1/\partial q^3 & \partial x^2/\partial q^3 & \partial x^3/\partial q^3 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{i}}_2 \\ \hat{\mathbf{i}}_3 \end{bmatrix}$$

# Vector Calculus & General Coordinate Systems

## Fundamental Metric Tensor

In a unitary system, the square of the differential distance separating two infinitesimally spaced points is,

$$d\mathbf{r}\cdot d\mathbf{r}=(ds)^2=(\mathbf{e}_i\cdot\mathbf{e}_j)dq^i dq^j$$

Now define the components of the *fundamental metric tensor* as,

$$g_{ij}\equiv\mathbf{e}_i\cdot\mathbf{e}_j$$

Then,

$$d\mathbf{r}\cdot d\mathbf{r}=(ds)^2=g_{ij}dq^i dq^j$$

# Vector Calculus & General Coordinate Systems

In matrix format, the fundamental metric tensor is,

$$G = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}$$

Properties of the fundamental metric tensor:

1. Symmetric, i.e.,

   $$\mathbf{e}_i \cdot \mathbf{e}_j = \mathbf{e}_j \cdot \mathbf{e}_i \leftrightarrow g_{ij} = g_{ji}$$

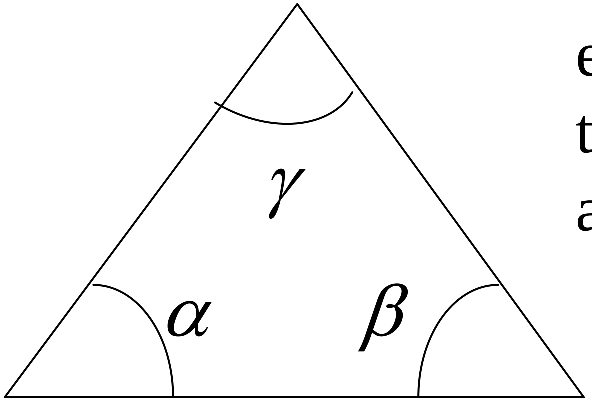
2. The *norm* (magnitude) of the unitary base vectors is,

   $$|\,\mathbf{e}_i\,| = (\mathbf{e}_i \cdot \mathbf{e}_i)^{1/2} = (g_{ii})^{1/2} \quad \text{(no summation)}$$

# Vector Calculus & General Coordinate Systems

3. Describes the curvature of the space,
   a) A *flat space* has no curvature and is called *Euclidean*. In this case, all the $g_{ij}$ components are constant.
   b) A *curved space* is called *Riemannian*  In this case, the $g_{ij}$ components are not constant.  An example is *Lobachevskian* space.  This space has hyperbolic curvature.

Euclidean



We can compare these two spaces by looking at the geometry of a triangle in each.  For Euclidean geometry, we know the sum of the interior angles of a triangle is always 180°.

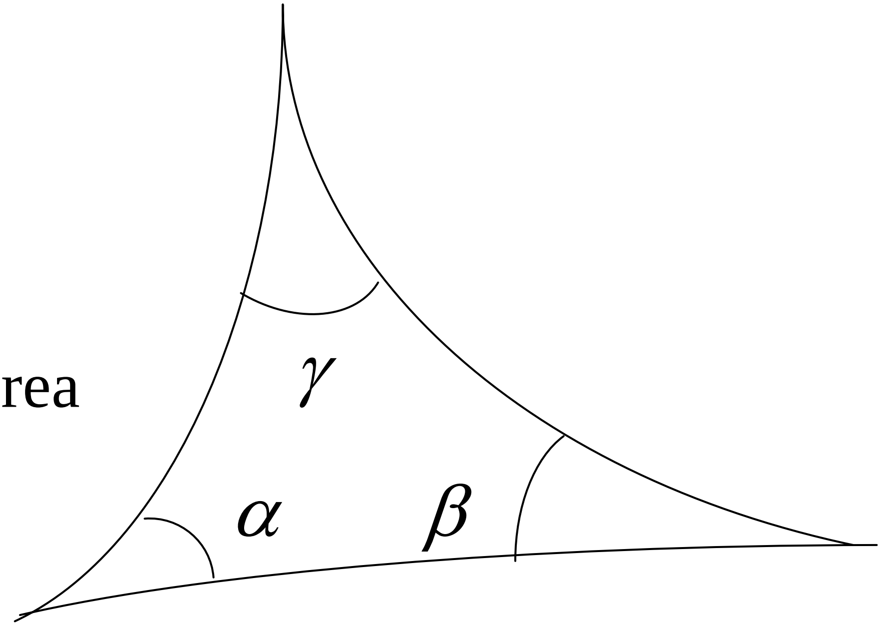
$$\alpha + \beta + \gamma = 180^\circ$$

# Vector Calculus & General Coordinate Systems

In Lobachevskian geometry, that sum is always *less* than 180°, the difference being proportional to the area of the triangle. (Penrose, Roger, *The Emperor’s New Mind*, p. 156).

Lobachevskian

$$\alpha + \beta + \gamma \neq 180° = \text{const} \times \text{area}$$



# Vector Calculus & General Coordinate Systems

Example: Find the unitary basis vectors and components of the fundamental metric tensor for elliptic-cylindrical, coordinates defined by the following inverse transformation (*a* = constant):

$$x^1 = a\cosh q^1 \cos q^2, \quad x^2 = a\sinh q^1 \sin q^2, \quad x^3 = q^3$$

In terms of the Cartesian basis, the unitary basis is,

$$\mathbf{e}_1 = \frac{\partial x^i}{\partial q^1}\hat{\mathbf{i}}_i = a\sinh q^1 \cos q^2 \hat{\mathbf{i}}_1 + a\cosh q^1 \sin q^2 \hat{\mathbf{i}}_2$$

$$\mathbf{e}_2 = \frac{\partial x^i}{\partial q^2}\hat{\mathbf{i}}_i = -a\cosh q^1 \sin q^2 \hat{\mathbf{i}}_1 + a\sinh q^1 \cos q^2 \hat{\mathbf{i}}_2$$

$$\mathbf{e}_2 = \frac{\partial x^i}{\partial q^3}\hat{\mathbf{i}}_i = \hat{\mathbf{i}}_3$$

# Vector Calculus & General Coordinate Systems

Components of the fundamental metric tensor are:

$$g_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1 = a^2[\sinh^2(q^1)\cos^2(q^2) + \cosh^2(q^1)\sin^2(q^2)$$

$$= g_{22}$$

$$g_{33} = 1, \quad g_{12} = g_{21} = g_{13} = g_{31} = 0$$

## Components and Bases

Recall,

$$\mathbf{a} = (\mathbf{a} \cdot \mathbf{e}^i)\mathbf{e}_i = (\mathbf{a} \cdot \mathbf{e}_i)\mathbf{e}^i$$

Now set $\mathbf{a} = \mathbf{e}^j$ ,

$$\mathbf{e}_j = (\mathbf{e}_j \cdot \mathbf{e}_i)\mathbf{e}^i$$

With the definition for the components of *G*, we have,

# Vector Calculus & General Coordinate Systems

$g^{ij} \equiv \mathbf{e}^i \cdot \mathbf{e}^j$ contravariant component of the fundamental metric.

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Then according to the cogredient and contragredient transformation laws *raising and lowering of the indices* is accomplished with the following,

$\mathbf{e}^j = g^{ij}\,\mathbf{e}_i$ and $a^j = g^{ij}\,a_i$

$\mathbf{e}_j = g_{ij}\,\mathbf{e}^i$ and $a_j = g_{ij}\,a^i$ .

Note that when dealing with a unitary basis, cogredient components and vectors are referred to as *covariant* components and contragredient components and vectors are referred to as *contravariant* components.

# Vector Calculus & General Coordinate Systems

Also, if we "dot" both sides of the $\mathbf{e}_j$ transformation equation,

$(\mathbf{e}_j = g_{ij}\, \mathbf{e}^i) \cdot \mathbf{e}^k$,

then we get the neat result

$$\delta_j^k = g_{ij} g^{ik} \qquad (4)$$

For this relation, note the sum over $i$, e.g.,

$$\delta_1^1 = g_{11} g^{11} + g_{21} g^{21} + g_{31} g^{31} = 1,$$

$$\delta_1^2 = g_{11} g^{12} + g_{21} g^{22} + g_{31} g^{32} = 0.$$

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# Vector Calculus & General Coordinate Systems

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$$\begin{tikzcd}
 & g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \\
\mathbf{e}_i \arrow[ur] \arrow[dr] & \\
 & \mathbf{e}^i = \dfrac{\mathbf{e}_j \times \mathbf{e}_k}{[\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3]} \rightarrow g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j \qquad (ijk \text{ cyclic})
\end{tikzcd}$$

# Vector Calculus & General Coordinate Systems

The cross product step is avoided by using the linear transformation

$$\xcancel{(\mathbf{e}_j = g_{ij}\,\mathbf{e}^i)\cdot\mathbf{e}^k} \tag{5}$$

~~or~~ in matrix notation,

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} \mathbf{e}^1 \\ \mathbf{e}^2 \\ \mathbf{e}^3 \end{bmatrix} \tag{6}$$

and

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix} \tag{7}$$

# Vector Calculus & General Coordinate Systems

To determine the $\mathbf{e}^j$ in terms of the $\mathbf{e}_i$ the matrix equation (6) must be inverted. Let,

$g \equiv |g_{ij}| \equiv \det G$ ,

$M_{ij}$ = minor of $g_{ij}$ ,

$C_{ij} \equiv (-1)^{i+j} M_{ij}$ = cofactor of $g_{ij}$ (no summation).

Employing Cramer's rule,

$$\mathbf{e}^1 = \begin{vmatrix} \mathbf{e}_1 & g_{12} & g_{13} \\ \mathbf{e}_2 & g_{22} & g_{23} \\ \mathbf{e}_3 & g_{32} & g_{33} \end{vmatrix} = \frac{\mathbf{e}_1 M_{11} - \mathbf{e}_2 M_{21} + \mathbf{e}_3 M_{31}}{g} = \frac{\mathbf{e}_i C_{i1}}{g}$$

We obtain similar expressions for $\mathbf{e}^2$ and $\mathbf{e}^3$. In general then

$$\mathbf{e}^j = \frac{C_{ij}}{g} \mathbf{e}_i .$$

# Vector Calculus & General Coordinate Systems

Continuing in matrix format, you will probably recognize where this is leading, from the previous section on linear algebra. Since, $g_{ij} = g_{ji}$, the fundamental metric tensor is symmetric and $C_{ij} = C_{ji}$, then,

$$\left[\mathbf{e}^j\right] = \frac{1}{g}\left[C_{ij}\right]^T\left[\mathbf{e}_i\right] \quad \rightarrow \quad \left[\mathbf{e}^j\right] = G^{-1}\left[\mathbf{e}_i\right]$$

so

$$\frac{1}{g}\left[C_{ij}\right]^T = G^{-1}.$$

We designate the elements of $G^{-1}$ with superscripts, i.e.,

$$G^{-1} = \left[g^{ij}\right].$$

# Vector Calculus & General Coordinate Systems

So what have we accomplished with all this? If $G = [g_{ij}]$ is known, we can use linear transformations and the rules of linear algebra to determine the dual basis and covariant components without formulae that involve cross products. In fact, knowing what we now know about systems of linear equations we could have anticipated this result from the matrix representation of Eq. (6), i.e.,

$$\left[\mathbf{e}_i\right] = G\left[\mathbf{e}^j\right] \quad \rightarrow \quad \left[\mathbf{e}^j\right] = G^{-1}\left[\mathbf{e}_i\right].$$

Another thing to note is the result in Eq. (4) is also anticipated since, in matrix notation, the Kronecker delta is the unit matrix,

# Vector Calculus & General Coordinate Systems

$$\left[\delta^i_j\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that the product in Eq. (4) is just,

$$\delta^k_j = g_{ij} g^{ik} \quad \leftrightarrow \quad I = GG^{-1}.$$

# Vector Calculus & General Coordinate Systems

## The General Permutation Symbol

In the Cartesian system, the cross product is well defined analytically and geometrically. What about general coordinates?

We define the general permutation symbol by the operation

$\mathbf{e}_i \times \mathbf{e}_j = \mathcal{E}_{ijk}\mathbf{e}^k$ (for a right-handed system)

where

$$\mathcal{E}_{ijk} = \mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k \equiv [\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3].$$

Using $\det(g_{ij}) = g = [\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3]^2$, we then write

$$\mathcal{E}_{ijk} = \sqrt{g}\,\varepsilon_{ijk} \quad \text{and} \quad \mathcal{E}^{ijk} \equiv \frac{1}{\sqrt{g}}\varepsilon_{ijk}.$$

# Vector Calculus & General Coordinate Systems

## Physical Components of a Vector

Recall a physical component of a vector is defined by

$\hat{a}^i \hat{\mathbf{e}}_i = a^i \mathbf{e}_i$ (no summation).

Then,

$$\hat{a}^i \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i = a^i \mathbf{e}_i \cdot \hat{\mathbf{e}}_i$$

$$\hat{a}^i = a^i \mathbf{e}_i \cdot \frac{\mathbf{e}_i}{|\mathbf{e}_i|} \quad \rightarrow \quad \hat{a}^i = a^i \frac{g_{ii}}{\sqrt{g_{ii}}} \quad \text{(no summation).}$$

Therefore, the physical component, in terms of the contravariant and covariant components is,

$$\hat{a}^i = a^i \sqrt{g_{ii}} \text{ and similarly } \hat{a}_i = a_i \sqrt{g^{ii}} \quad \text{(no summation).}$$

# Vector Calculus & General Coordinate Systems

## Orthogonal Curvilinear Coordinate Systems

Because of the many conveniences of orthogonal systems, most space-coordinate systems used in engineering analysis are orthogonal. Many of these systems are also curvilinear systems, in particular, the spherical and cylindrical systems with which you are familiar. In this section we will look at *orthogonal curvilinear systems* and how they relate to our original Cartesian system.

<u>Scale Factors</u>

Define the *scale factors*

$$h_1 = |\mathbf{e}_1| = \sqrt{g_{11}}, \quad h_2 = |\mathbf{e}_2| = \sqrt{g_{22}}, \quad h_3 = |\mathbf{e}_3| = \sqrt{g_{33}}.$$

# Vector Calculus & General Coordinate Systems

## Orthogonal Curvilinear Coordinate Systems

With these definitions, then,

$$\mathbf{e}_1 = h_1\hat{\mathbf{e}}_1, \quad \mathbf{e}_2 = h_2\hat{\mathbf{e}}_2, \quad \mathbf{e}_3 = h_3\hat{\mathbf{e}}_3$$

For a general curvilinear system, we earlier showed that a differential displacement is written as,

$$d\mathbf{r} = dq^1\mathbf{e}_1 + dq^2\mathbf{e}_2 + dq^3\mathbf{e}_3.$$

Now using the scale factors,

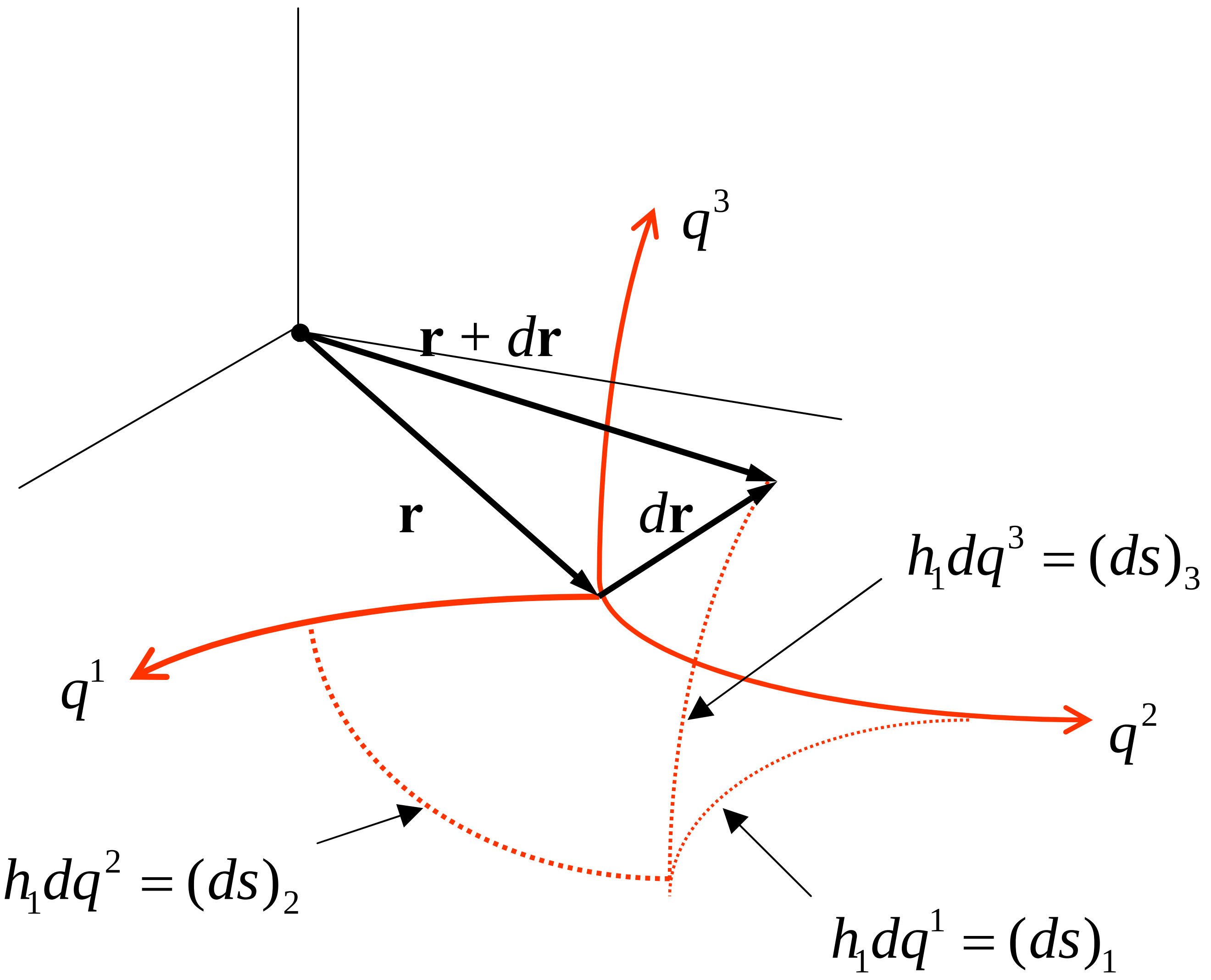
$$d\mathbf{r} = (h_1 dq^1)\hat{\mathbf{e}}_1 + (h_2 dq^2)\hat{\mathbf{e}}_2 + (h_3 dq^3)\hat{\mathbf{e}}_3.$$

So, for the arclength, the differential distances are

$$d\mathbf{r}\cdot d\mathbf{r} = (ds)^2 = (h_1 dq^1)^2 + (h_2 dq^2)^2 + (h_3 dq^3)^2$$

$$\rightarrow ds_1 = h_1 dq^1, \quad ds_2 = h_2 dq^2, \quad ds_3 = h_3 dq^3$$

# Vector Calculus & General Coordinate Systems



# Vector Calculus & General Coordinate Systems

*The scale factors scale the $q^j$ to the appropriate magnitude and dimension for an orthogonal curvilinear system.*

In terms of the original Cartesian system,

$$h_1 \equiv \left|\frac{\partial \mathbf{r}}{\partial q^1}\right| = \left[\left(\frac{\partial x^1}{\partial q^1}\right)^2 + \left(\frac{\partial x^2}{\partial q^1}\right)^2 + \left(\frac{\partial x^3}{\partial q^1}\right)^2\right]^{1/2} ,$$

$$h_2 \equiv \left|\frac{\partial \mathbf{r}}{\partial q^2}\right| = \left[\left(\frac{\partial x^1}{\partial q^2}\right)^2 + \left(\frac{\partial x^2}{\partial q^2}\right)^2 + \left(\frac{\partial x^3}{\partial q^2}\right)^2\right]^{1/2} ,$$

$$h_3 \equiv \left|\frac{\partial \mathbf{r}}{\partial q^3}\right| = \left[\left(\frac{\partial x^1}{\partial q^3}\right)^2 + \left(\frac{\partial x^2}{\partial q^3}\right)^2 + \left(\frac{\partial x^3}{\partial q^3}\right)^2\right]^{1/2} .$$

# Vector Calculus & General Coordinate Systems

## Differential Volume Element

In many applications, especially finite-volume and finite-element methods, you often must determine the volume of a differential element. For instance, a finite-volume form of the mass conservation equation in fluid mechanics requires a computation of the flux of mass through the boundaries, which must balance the creation of mass inside the volume. In most applications, the differential cell (volume) is of some variable shape determined by a curvilinear coordinate system. Here we introduce a general expression for determining a differential volume.

Recall how the scalar triple product is related to the volume of a parallelepiped (with appropriate sign):

# Vector Calculus & General Coordinate Systems

$[\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3]$ = volume of parallelepiped (with appropriate sign.

In general,

$$dV = \mathbf{ds}_1 \cdot \mathbf{ds}_2 \times \mathbf{ds}_3$$

$$= dq^1\mathbf{e}_1 \cdot dq^2\mathbf{e}_2 \times dq^3\mathbf{e}_3$$

$$= dq^1 dq^2 dq^3 [\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3]$$

$$= dq^1 dq^2 dq^3 \sqrt{g} = dq^1 dq^2 dq^3 J$$

For an orthogonal curvilinear system,

$$dV = dq^1\mathbf{e}_1 \cdot dq^2\mathbf{e}_2 \times dq^3\mathbf{e}_3$$

$$= h_1 h_2 h_3 dq^1 dq^2 dq^3 [\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3]$$

$$= h_1 h_2 h_3 dq^1 dq^2 dq^3$$

# Vector Calculus & General Coordinate Systems

Finally, for the Cartesian system, the familiar result

$$dV = dx^1 dx^2 dx^3 = dx\,dy\,dz$$

Note we can gain a bit of insight into the physical meaning of the Jacobian $J$. Combining the general expression for the differential volume element with that for the Cartesian system, we find,

$$J = \frac{dV}{dq^1 dq^2 dq^3} = \frac{dx\,dy\,dz}{dq^1 dq^2 dq^3}.$$

This shows that *the Jacobian of the transformation is the ratio of a differential volume in the Cartesian system to that of the general system*. You can also see (if you haven't already discovered this) how the Jacobian is related to the fundamental metric, .i.e., $J = \sqrt{g}$.