Elements of Linear Algebra

Our discussion of vector algebra is now focused on coordinate transformations. The subscript notation results in transformation relations described by a system of linear equations. This provides the impetus for us to revisit the basic concepts and operations from linear algebra.

Matrix Definitions and Basic Concepts

A *matrix* is a rectangular array of objects (*elements* that are numbers, functions, etc.) with its size indicated by the number of rows and columns, i.e., an $m \times n$ matrix A with m rows and n columns.

$$A \equiv [a_{ij}] \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{1m} & & & a_{mn} \end{bmatrix}$$

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For a *square matrix* $m = n$, the elements $a_{11}, a_{22}, \ldots, a_{mn}$ are the *main diagonal elements*.

Column matrix:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

Row matrix:

$$\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}$$

We will denote rectangular matrices with uppercase letters and column and row matrices (also referred to as vectors) with bold lowercase letters.

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A *submatrix* of a matrix A is obtained by deleting certain rows and/or columns of A .

Transpose: An interchange of the rows and columns, e.g.,

$$A \equiv [a_{ij}] \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{1m} & & & a_{mn} \end{bmatrix} \rightarrow A^T \equiv [a_{ji}] \equiv \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & \\ \vdots & & \ddots & \\ a_{1n} & & & a_{mn} \end{bmatrix}$$

A *diagonal matrix* is a square matrix with all off-diagonal elements zero; e.g.,

$$B = \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}$$

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A *symmetric matrix* is a square matrix with $[a_{ij}] = [a_{ji}]$, or $A = A^T$.

An *antisymmetric (skew-symmetric) matrix* is a square matrix with $a_{ij} = -a_{ji}$ or $A = -A^T$. Note that this definition requires the diagonal elements be zero.

Trace: sum of the diagonal elements.

Lower triangular matrix: A square matrix for which all elements above the diagonal are zero, e.g.,

$$L = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

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Upper triangular matrix: A square matrix for which all elements below the diagonal are zero, e.g.,

$$L = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

Unit matrix: A square matrix for which all diagonal elements are one and all off-diagonal elements are zero, e.g.,

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Each element is zero in the *zero matrix*, designated as 0.

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Rules for Addition and Scalar Multiplication

Addition is only defined for matrices of the same size, e.g.,

$C = A + B \rightarrow [c_{ij}] = [a_{ij}] + [b_{ij}]$.

Rules:

1. $A + B = B + A$
2. $A + B + C = (A + B) + C = A + (B + C)$
3. $A + 0 = A$
4. $A + (-A) = 0$

Multiplication of a scalar and a matrix: Each element of the matrix is multiplied by the scalar, e.g., if k is a scalar then,

$kA = [ka_{ij}]$.

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Matrix Multiplication

One of the motivations for defining matrix multiplication is that it allows us to write systems of linear equations in a more compact form. It also allows us to apply certain other matrix operations and manipulations that allow us to solve linear systems.

The product of two matrices *A* and *B* is defined by the operation,

$$C = AB \rightarrow [c_{ij}] = \sum_{k=1}^{n} a_{ik} b_{kj},$$

Where *n* is the column dimension of *A* and the row dimension of *B*, e.g.,

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$$C = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -3 \\ 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -3 \\ 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1\times 1 + 0\times 2 + 2\times 1 & \bullet \\ \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -3 \\ 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1\times 2 + 0\times(-3) + 2\times 3 \\ \bullet & \bullet \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -3 \\ 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 8 \\ -1\times 1 + 3\times 2 + 4\times 1 & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -3 \\ 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 8 \\ 9 & -1\times 2 + 3\times (-3) + 4\times 3 \end{bmatrix}$$

$$\rightarrow C = \begin{bmatrix} 3 & 8 \\ 9 & 1 \end{bmatrix}$$

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Note the matrix C has the same number of rows as the *prefactor A* and the same number of columns as the *postfactor B*.

To illustrate the importance of the order of the prefactor and the postfactor, here is an example of the product of a row vector and a column vector,

$$C = \begin{bmatrix} 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = [19],$$

but interchanging the order gives,

$$C = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}.$$

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This illustrates one of the most important rules of matrix multiplication: *In general, matrix multiplication is not commutative*, i.e.,

1. $AB \neq BA$, in general.

The prefactor A *premultiplies* the postfactor B on the left-hand side of the inequality and A *postmultiplies* B on the right-hand side of the inequality.

An additional consequence of the rules of matrix multiplication is that,

2. $AB = 0$ does not always imply $A = 0$, or $B = 0$, or $BA = 0$.

Additional rules are:

3. $ABC = A(BC) = (AB)C$
4. $(A + B)C = AC + BC$ and $C(A + B) = CA + CB$

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5. $AI = IA = A$, where I is the identity (unit) matrix.
6. $(AB)^T = B^T A^T$
7. If A is a square matrix, the powers of A are defined by $A^2 = AA$, $A^3 = AA^2$.

Matrices and Linear Transformations

The multiplication of a square matrix A and a column vector $\mathbf{x}$ transforms the vector into another column vector $\mathbf{b}$, i.e., matrix $\mathbf{A}$ is an *operator* that transforms vector $\mathbf{x}$ into vector $\mathbf{b}$,

$A\mathbf{x} = \mathbf{b}$.

With this result, let's now return to the vector transformation laws. In particular, the contragredient component transformation law is written in index notation as,

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$$\bar{a}^s = \beta^s_i a^i .$$

In n = 3 space, this is a system of three linear equations,

$$\bar{a}^1 = \beta^1_1 a^1 + \beta^1_2 a^2 + \beta^1_3 a^3$$

$$\bar{a}^2 = \beta^2_1 a^1 + \beta^2_2 a^2 + \beta^2_3 a^3$$

$$\bar{a}^3 = \beta^3_1 a^1 + \beta^3_2 a^2 + \beta^3_3 a^3$$

Using the rules of matrix multiplication, we can write the system as

$$\begin{bmatrix} \bar{a}^1 \\ \bar{a}^2 \\ \bar{a}^3 \end{bmatrix} = \begin{bmatrix} \beta^1_1 & \beta^1_2 & \beta^1_3 \\ \beta^2_1 & \beta^2_2 & \beta^2_3 \\ \beta^3_1 & \beta^3_2 & \beta^3_3 \end{bmatrix} \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix}$$

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This form allows us to introduce a number of special techniques for solving general systems of linear equations.

For the general system $A\mathbf{x} = \mathbf{b}$, the $(m \times n)$ matrix A is defined as the *coefficient matrix*; **x** and **b** are both column vectors. If all the elements of **b** are zero, the system is called *homogeneous*, otherwise, it is called *nonhomogeneous*.

The augmented matrix is defined by combining **b** and A as,

$$\tilde{A} = [a_{ij} \mid b_j] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & b_2 \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

This matrix contains all the information from **A** and **b**, and is easier to work with using the basic row and column operations we will discuss.

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<u>Classifying Linear Systems</u>

A linear system is…

1. *Overdetermined* if there are more equations than unknowns (may or may not have a solution).
2. *Determined* if $m = n$ (may or may not have a solution).
3. *Underdetermined* if there are more unknowns than equations (always has a solution).

Elementary Row Operations

1. Interchange of two rows (columns).
2. Multiplication of a row (column) by a nonzero scalar.
3. Replace a row (column) with scalar multiple of itself and another row (column).

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Two linear systems are *row equivalent* if one is obtainable from the other by a finite sequence of elementary operations. This is the condition that guarantees that the solution of a linear system modified by elementary operations is a solution of the original system.

The *rank* of a matrix or augmented matrix is designated r and is the number of nonzero rows after the matrix has been reduced to *echelon form*. The echelon form is the form of the augmented matrix after the last step of the *Gauss elimination*. (Note that one can show that A^T is the same rank as A.

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Existence and Uniqueness Conditions for Linear Systems

1. The linear system of *m* equations has a solution if and only if is of the same rank as *A*.
2. If $r = n$, the system has a unique solution.
3. If $r < n$, there are an infinite number of solutions where *r* of the unknowns can be expressed in terms of the remaining $n - r$ unknowns.

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Gaussian Elimination

For Gaussian elimination, the elementary row operations are used to reduce the system of equations to the form shown here (for $r \leq m$ and $a_{11} \neq 0$, $c_{22} \neq 0, \ldots, k_{rr} \neq 0$).

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
c_{22}x_2 + \cdots + c_{2n}x_n &= b_2 \\
&\vdots \\
k_{rr}x_r + \cdots + k_{rn}x_n &= b_r \\
0 &= \bar{b}_{r+1} \\
&\vdots \\
0 &= \bar{b}_m
\end{aligned}
$$

This is the echelon form just defined and the rules are essentially those of the general case,

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1. No solution if $r < m$ and one of the numbers $\bar{b}_{r+1}, \cdots, \bar{b}_m$ is not zero.
2. One unique solution if $r = n$ and $\bar{b}_{r+1}, \cdots, \bar{b}_m$, if present, are zero.
3. Infinitely many solutions if $r < n$.

<u>Example</u>: Use Gaussian elimination to show that unique to the 3×3 system:

$$\begin{aligned} 3x_1 + x_2 - x_3 &= 2 \\ x_1 + 4x_2 + x_3 &= 12 \\ 2x_1 + x_2 + 2x_3 &= 10 \end{aligned} \qquad \text{is} \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

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Solution:

1. Write the augmented matrix:

$$\begin{bmatrix} 3 & 1 & -1 & 2 \\ 1 & 4 & 1 & 12 \\ 2 & 1 & 2 & 10 \end{bmatrix}$$

2. Row operations, row 1 (R_1) is the *pivot row* and a_{11} = 3 the *pivot element* used to create zeros in the first column,

$$R_2 - \frac{1}{3} R_1 \to R_2 \quad \text{and} \quad R_3 - \frac{2}{3} R_1 \to R_3$$

$$\begin{bmatrix} 3 & 1 & -1 & 2 \\ 0 & 11/3 & 4/3 & 34/3 \\ 0 & 1/3 & 8/3 & 26/3 \end{bmatrix}$$

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R_2 is now the pivot row and a_{22} the pivot element,

$$R_3 - \frac{3}{33}R_2 \rightarrow R_3$$

$$\begin{bmatrix} 3 & 1 & -1 & 2 \\ 0 & 11/3 & 4/3 & 34/3 \\ 0 & 0 & 84/33 & 252/33 \end{bmatrix}$$

3. Now that the coefficient matrix is in *upper triangular form* (also in *echelon form*), use back substitution to determine the unknowns x_i,

$$R_3 : \frac{84}{33}x_3 = \frac{252}{33} \quad \rightarrow x_3 = 3$$

$$R_2 : \frac{11}{3}x_2 + \frac{4}{3}(3) = \frac{34}{3} \quad \rightarrow x_2 = 2$$

$$R_1 : 3x_1 + (2) - (3) = 2 \quad \rightarrow x_1 = 1$$

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Determinant

For a square ($n \times n$) matrix A, the *determinant* of A is a special number that, according to Reddy & Rasmussen, in some sense measures the "size" of A and indicates whether or not A is "singular" (a term we will define later).

We introduce the computation of a general $n \times n$ determinant by showing the computation procedure for low-order determinants. For a 1×1 matrix, by definition,

$\det A = |A| = |a_{11}| = a_{11}$.

For a 2×2 matrix A, the determinant is defined according to,

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

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A 3×3 determinant is expanded in terms of the 2×2,

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Higher-order determinants are defined as a sum of multiples of determinants of $(n-1)\times(n-1)$ matrices. These $(n-1)\times(n-1)$ matrices are the sum of determinants of $(n-2)\times(n-2)$ matrices and so on. Computationally, we reduce the $n\times n$ determinant until we can evaluate 3×3 or 2×2 determinants, for which we have the previous explicit formulas.

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For $n \geq 2$, define the *minor of* a_{ij} as M_{ij}

M_{ij} = determinant of the $(n-1)\times(n-1)$ submatrix formed by deleting row i and column j of A.

And the *cofactor of* a_{ij} as $C_{ij} = (-1)^{i+j} M_{ij}$

We can now define the determinant of a $n\times n$ ($n \geq 2$) matrix A as,

$$\det A = |A| = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} M_{kj} \rightarrow \text{cofactor expansion by row } k$$

$$\det A = |A| = \sum_{i=1}^{n} (-1)^{i+k} a_{ik} M_{ik} \rightarrow \text{cofactor expansion by column } k$$

Note that one obtains the same result whether the expansion is by rows *or* columns!

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The *adjoint matrix* is the transpose of the matrix obtained by replacing each of the elements by its corresponding cofactor, e.g., for a $n = 3$ case,

$$\mathrm{Adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

We will later make use of the adjoint matrix when computing the inverse of a matrix.

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As shown earlier, the vector (cross) product of two vectors **a** and **b** can be expressed in terms of a determinant; the scalar triple product of three vectors **a**, **b**, and **c** can also be represented by a determinant,

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \text{and} \quad \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Additional properties of determinants:

1. det (AB) = det A det B
2. det A^T = det A
3. det (kA) = k^n det A

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4. If A' is a matrix obtained from A by multiplying a row (or column) of A by a scalar k, then det $A' = k$ det A.
5. If any two rows (or columns) of a determinant are interchanged, the value of the determinant is multiplied by –1.
6. If two rows (or columns) of A are proportional, the value of the determinant is zero.
7. The value of a determinant is unchanged by adding a multiple of one row (or column) to another.
8. A matrix is *singular* if and only if (iff) its determinant is zero. As mentioned in property 6., this indicates that two rows (or columns) are *linearly dependent*.

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Inverse of a Matrix

For the $n \times n$ matrix A, the *inverse* of A is denoted A^{-1} and is defined by

$$AA^{-1} = A^{-1}A = I$$

If a square matrix A has an inverse, that inverse A^{-1} is *unique* and A is *nonsingular*. If A does not have an inverse, it is said to be *singular*.

If one can compute the inverse of the coefficient matrix of a linear system of equations $A\mathbf{x} = \mathbf{b}$, then the solution to the system is obtained using,

$$AA^{-1}\mathbf{x} = I\mathbf{x} = A^{-1}\mathbf{b} \quad \rightarrow \mathbf{x} = A^{-1}\mathbf{b}$$

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The inverse A^{-1} can be computed using the formula,

$$A^{-1} = \frac{1}{\det A}\text{Adj}(A)$$

<u>Example</u>: Compute A^{-1} from the previous example,

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 4 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

1. First compute cofactors:

$$C_{11} = \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 7, \quad C_{12} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0, \quad C_{13} = \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} = -7, \cdots$$

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2. Construct adjoint matrix and compute determinant (expanded by first row):

$$\mathrm{Adj}(A) = \begin{bmatrix} 7 & 0 & -7 \\ -3 & 8 & -1 \\ 5 & -4 & 11 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 & -5 \\ 0 & 8 & -4 \\ -7 & -1 & 11 \end{bmatrix}$$

$$\det A = (3)\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} - (1)\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} = 28$$

3. Compute A^{-1}

$$A^{-1} = \frac{\mathrm{Adj}(A)}{\det A} = \begin{bmatrix} 1/4 & -3/28 & 5/28 \\ 0 & 2/7 & -1/7 \\ -1/4 & -1/28 & 11/28 \end{bmatrix} \Leftarrow$$

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Gauss-Jordan elimination for computing A^{-1}

1. Write the augmented matrix $[A|I]$

$$[A\,|\,I]=\left[\begin{array}{ccc|ccc} 3 & 1 & -1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

2. Row operations

$$R_2-\frac{1}{3}R_1\to R_2 \quad \text{and} \quad R_3-\frac{2}{3}R_1\to R_3$$

$$\left[\begin{array}{ccc|ccc} 3 & 1 & -1 & 1 & 0 & 0 \\ 0 & 11/3 & 4/3 & -1/3 & 1 & 0 \\ 0 & 1/3 & 8/3 & -2/3 & 0 & 1 \end{array}\right]$$

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$$R_3 - \frac{3}{33} R_2 \rightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 3 & 1 & -1 & 1 & 0 & 0 \\ 0 & 11/3 & 4/3 & -1/3 & 1 & 0 \\ 0 & 0 & 84/33 & -7/11 & -1/11 & 1 \end{array}\right]$$

3. Eliminate the upper elements on the LHS of the partition to create the identity matrix.

$$\frac{3}{11} R_2 \rightarrow R_2 \quad \text{and} \quad \frac{33}{84} R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 3 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4/11 & -1/11 & 3/11 & 0 \\ 2 & 0 & 1 & -1/4 & -1/28 & 11/28 \end{array}\right]$$

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$$R_2 - \frac{4}{11} R_3 \rightarrow R_2 \quad \text{and} \quad R_1 + R_3 \rightarrow R_1$$

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 3/4 & -1/28 & 11/28 \\ 0 & 1 & 0 & 0 & 2/7 & -1/7 \\ 0 & 0 & 1 & -1/4 & -1/28 & 11/28 \end{array}\right]$$

$$R_1 - R_2 \rightarrow R_1 \quad \text{and} \quad R_1 / 3 \rightarrow R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -3/28 & 5/28 \\ 0 & 1 & 0 & 0 & 2/7 & -1/7 \\ 0 & 0 & 1 & -1/4 & -1/28 & 11/28 \end{array}\right]$$

$$\rightarrow A^{-1} = \begin{bmatrix} 1/4 & -3/28 & 5/28 \\ 0 & 2/7 & -1/7 \\ -1/4 & -1/28 & 11/28 \end{bmatrix} \quad \Leftarrow$$

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Cramer's Rule

Although not as efficient as other methods for computing the solution of an $n\times n$ system of linear equations, Cramer's rule is useful when solving differential and partial differential equations (e.g., the method of characteristics).

If the determinant of the coefficient matrix, $D = \det A$, of an $n\times n$ system of linear equations, $A\mathbf{x} = \mathbf{b}$, is not zero, the unique solution is given by,

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}, \quad \cdots, \quad x_n = \frac{D_n}{D}$$

where D_i is the determinant with the *i-th* column of A replaced by $\mathbf{b}$.

As an example, we apply Cramer's rule to the earlier numerical example solved with Gauss elimination,

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$$\begin{aligned} 3x_1 + x_2 - x_3 &= 2 \\ x_1 + 4x_2 + x_3 &= 12 \\ 2x_1 + x_2 + 2x_3 &= 10 \end{aligned}$$

1. Compute determinants:

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 4 & 1 \\ 2 & 1 & 2 \end{bmatrix} \qquad \det A = 3\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} = 28$$

$$D_1 = \begin{vmatrix} 2 & 1 & -1 \\ 12 & 4 & 1 \\ 10 & 1 & 2 \end{vmatrix} = 28, \quad D_2 = \begin{vmatrix} 3 & 2 & -1 \\ 1 & 12 & 1 \\ 2 & 10 & 2 \end{vmatrix} = 56, \quad D_3 = \begin{vmatrix} 3 & 1 & 2 \\ 1 & 4 & 12 \\ 2 & 1 & 10 \end{vmatrix} = 84$$

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2. Solution:

$$x_1 = \frac{D_1}{D} = 1, \quad x_2 = \frac{D_2}{D} = 2, \quad x_3 = \frac{D_3}{D} = 3 \quad \Leftarrow$$

Note that it may seem that Cramer's rule did not require substantially more arithmetic operations than Gaussian elimination—however it did. It generally requires $O(n^3)$ basic arithmetic operations to solve a system of n-equations using Gauss or Gauss-Jordan elimination, but $O(n^4)$ basic arithmetic operations for Cramer's rule. For our case this is $O(27)$ compared to $O(81)$ operations. In terms of software packages, as far as I know, Cramer's rule is never used to solve systems of linear equations.