Calculus of Variations & Variational Methods

The calculus of variations is used to obtain extrema of expressions involving unknown functions called functionals. Applications range from simple geometric problems to finite-element methods to optimization theory.

References:

Ewing, G. M., *Calculus of Variations with Applications*, Dover, 1985.

Hildebrand, F. B., *Methods of Applied Mathematics, 2nd Ed.*, Dover, 1992.

Reddy & Rasmussen, *Advanced Engineering Analysis*, Kreiger 1990.

Weinstock, R., *Calculus of Variations with Applications to Physics and Engineering*, Dover.

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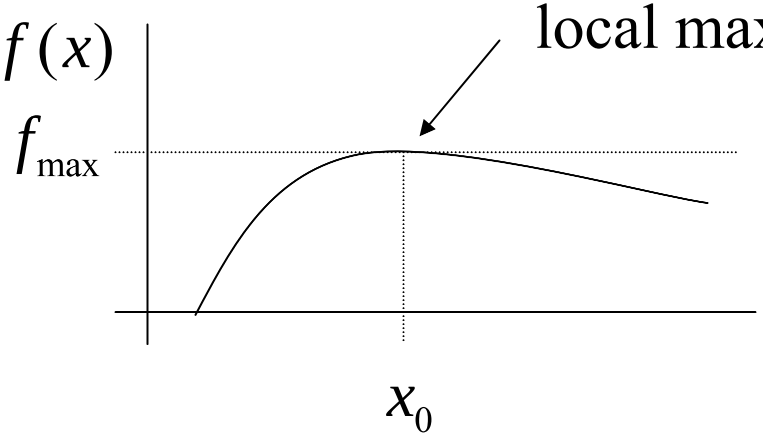
Unconstrained Minimization

In preparation for an introduction to the calculus of variations, recall maxima, minima (extrema), and inflections of functions from the differential calculus.

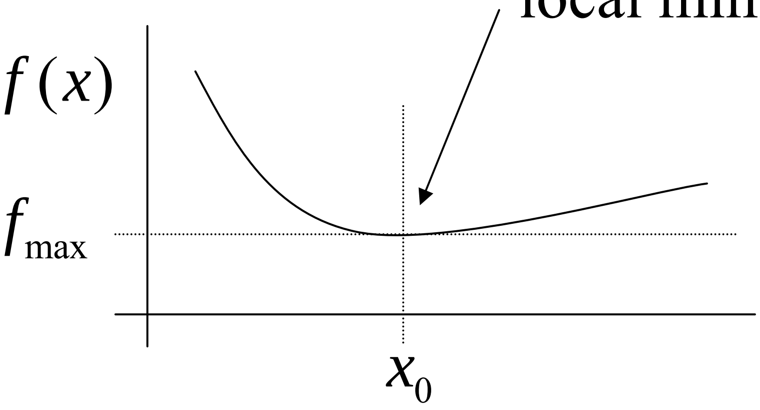


$$\frac{df}{dx} = 0$$

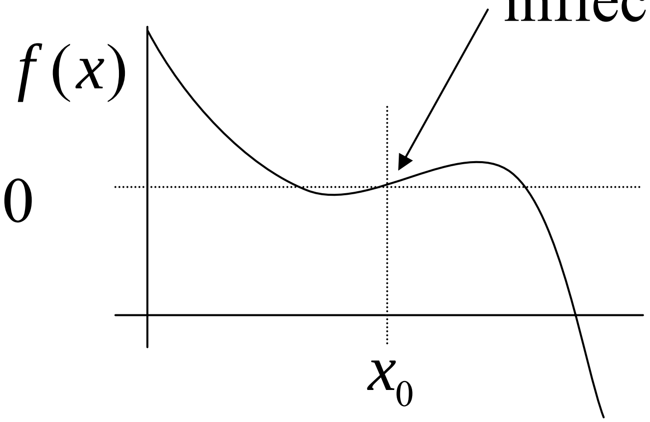
$$x_0 = \text{ critical (stationary) point}$$

$$f_{\max} = f(x_0)$$

$$= \text{ critical value}$$



$$\frac{df}{dx} = 0$$



$$\frac{df}{dx} = 0$$

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$$\left.\frac{df}{dx}\right|_{x=x_0} = 0, \text{ necessary condition for extremum}$$

$$\left.\frac{df}{dx}\right|_{x=x_0} = 0 \quad \begin{cases} \left.\dfrac{d^2 f}{dx^2}\right|_{x_0} < 0 \rightarrow \text{local max} \\ \left.\dfrac{d^2 f}{dx^2}\right|_{x_0} = 0 \rightarrow \text{ inflection} \\ \left.\dfrac{d^2 f}{dx^2}\right|_{x_0} > 0 \rightarrow \text{local min} \end{cases}$$

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Given $f(x, y)$, a necessary condition for a minimum at (x_0, y_0) is

$$df\Big|_{(x_0,y_0)} = \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right)\Bigg|_{(x_0,y_0)} = d\mathbf{r} \cdot \nabla f\Big|_{(x_0,y_0)} = 0$$

Since x and y are independent variables, dx and dy are independent so,

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

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Constrained Minimization

Now minimize *f* with a constraint, e.g.,

$$G(x,y)=0 \tag{1}$$

<u>Lagrange Multiplier Method</u>

From (1),

$$\frac{\partial G}{\partial x}dx+\frac{\partial G}{\partial y}dy=d\mathbf{r}\cdot\nabla G=0$$

We introduce the modified function with no constraints

$$F_L(x,y,\lambda)\equiv f(x,y)+\lambda G(x,y) \tag{2}$$

and set

$$dF_L=\frac{\partial F_L}{\partial x}dx+\frac{\partial F_L}{\partial y}dy+\frac{\partial F_L}{\partial \lambda}d\lambda=0 \tag{3}$$

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Example:

Find the stationary (critical) point of the function

$$f(x,y) = 2x^2 + y^2 - 8x + y + 1$$

with the constraint $2x - y = 0$

Sol'n:

$$F_L(x,y,\lambda) = (2x^2 + y^2 - 8x + y + 1) + \lambda(2x - y)$$

$$\left.\begin{aligned} \frac{\partial F_L}{\partial x} &= 4x - 8 + 2\lambda = 0, \\ \frac{\partial F_L}{\partial y} &= 2y + 1 - \lambda = 0, \\ \frac{\partial F_L}{\partial \lambda} &= 2x - y = 0. \end{aligned}\right\} \rightarrow x = 0.5, \quad y = 1.0, \quad \lambda = 3.0$$

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Calculus of Variations: Functionals and Euler Equations

The focus of the calculus of variations is the determination of maxima and minima of expressions that involve unknown functions. Here we look at a few classical problems to introduce some of the concepts.

The Brachistochrone (the one that started it all):

Weinstock gives the problem as it was originally stated by John Bernoulli in 1696: "Given two points *A* and *B* in a vertical plane, to find for the moveable particle *M*, the path *AMB*, descending along which by its own gravity, the beginning to be urged from point *A*, it may in the shortest time reach the point *B*."

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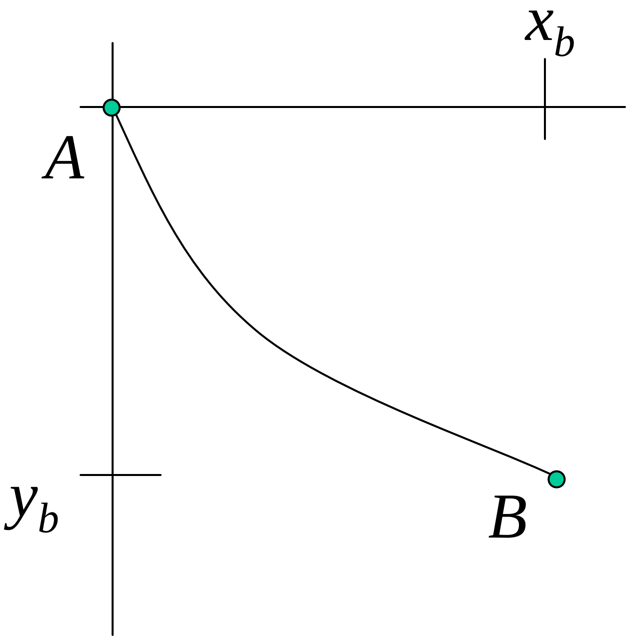
Reddy & Rasmussen state in engineering terms: “Design a chute between two points *A*: (0,0) and *B*: (x_b,y_b) in a vertical plane such that a material particle, sliding without friction under its own weight, travels from point *A* to point *B* along the chute in the shortest time.



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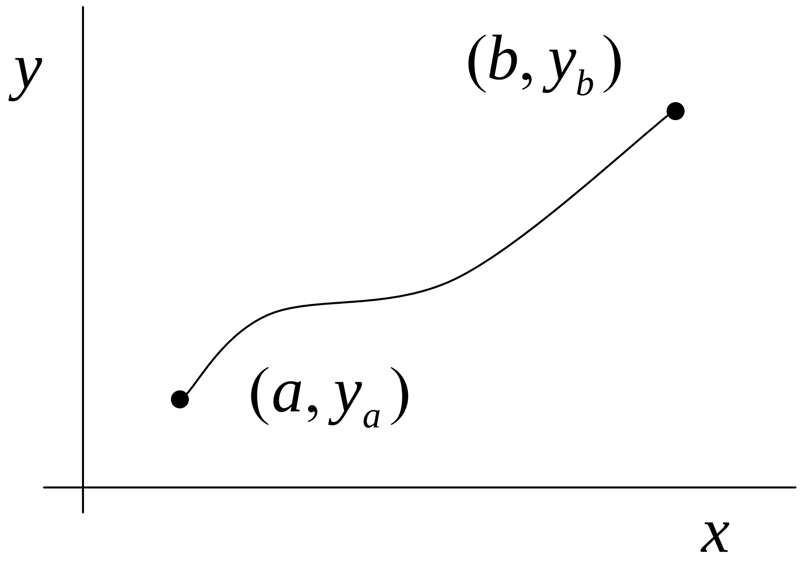
Other Classic Problems

Geodesic Problem

What is the curve of minimum length that connects two points?

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$y(a) = y_a, \quad y(b) = y_b$$



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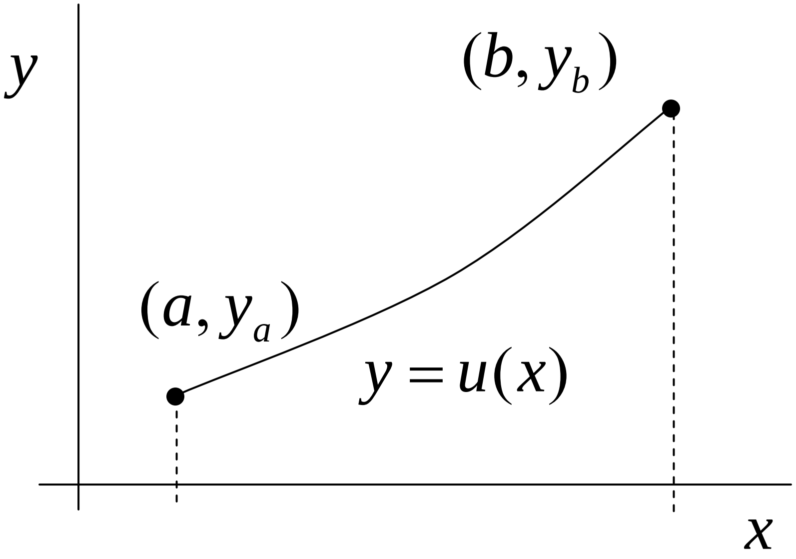
Minimum Surface of Revolution

What is the curve of minimum length that connects two points?

$$S = 2\pi \int_a^b y \, ds$$

$$= 2\pi \int_a^b u \sqrt{1 + \left(\frac{du}{dx}\right)^2} \, dx$$

$$y(a) = y_a, \quad y(b) = y_b$$



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The Euler Equation

Each of these problems presents the problem of finding a continuously differentiable function $u(x)$ that minimizes the integral of the form

$$I(u) = \int_a^b F(x, u, u')\, dx \qquad (1)$$

And that satisfies the end conditions

$$u(a) = u_a, \quad u(b) = u_b$$

We now suppose that $u(x)$ is the minimizing function, then choose any continuously differentiable function $\eta(x)$ and create a one-parameter 'trial' function

$$y = \overline{u}(x) = u(x) + \alpha\eta(x)$$

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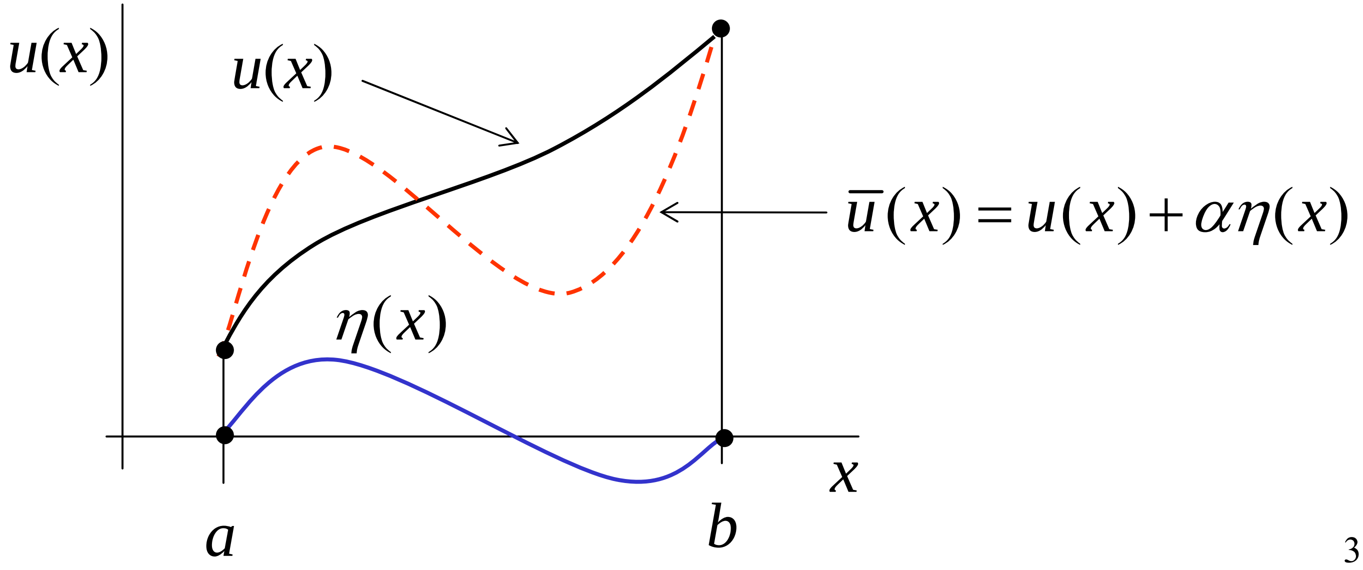
that vanishes at the end points $u = u_a$ and $u = u_b$. Then for any constant α,

$$\bar{u}(x) = u(x) + \alpha\eta(x) \quad \text{with } \eta(a) = \eta(b) = 0$$

satisfies the end conditions.

$$\bar{u}(a) = u_a, \quad \bar{u}(b) = u_b$$



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Now we substitute this function into the integral to be minimized

$$I(\alpha) = \int_a^b F(x, \overline{u}, \overline{u}')\, dx = \int_a^b F(x, u + \alpha\eta, u' + \alpha\eta')\, dx$$

where I is now a function of α once $u(x)$ and $\eta(x)$ are assigned. Since we have assumed that $u(x)$ is the minimizing function then $I(\alpha)$ is minimized when $\alpha = 0$. So, like the differential calculus, we now determine the stationary function (analogous to stationary points), but in this case we know in advance that $\alpha = 0$. So,

$$\frac{dI(\alpha)}{d\alpha} = 0$$

$$\frac{dI(\alpha)}{d\alpha} = \int_a^b \left(\frac{\partial F}{\partial \overline{u}} \frac{d\overline{u}}{d\alpha} + \frac{\partial F}{\partial \overline{u}'} \frac{d\overline{u}'}{d\alpha} \right) dx$$

$$= \int_a^b \left(\frac{\partial F}{\partial \overline{u}} \eta + \frac{\partial F}{\partial \overline{u}'} \eta' \right) dx$$

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Now when $\alpha \to 0$

$$\frac{dI(0)}{d\alpha} = \int_a^b \left(\frac{\partial F}{\partial u}\eta + \frac{\partial F}{\partial u'}\eta' \right) dx = 0.$$

Now we integrate the second term by parts,

$$\frac{dI(0)}{d\alpha} = \int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} \right) \right] \eta \, dx + \frac{\partial F}{\partial u'}\eta \bigg|_a^b = 0.$$

Since this must hold for all choices of $\eta(x)$, then

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} \right) = 0. \qquad (2)$$

This is the *Euler equation* or Euler-Lagrange equation.

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So, if $u(x)$ minimizes $I(u)$, it must satisfy the Euler equation. Solutions of the Euler equation are called *extremals* of the problem and an extremal that satisfies the end conditions is called a *stationary function*.

First Integrals

Equation (2) is written with partial derivatives that treat x, u, and u' as independent variables. The second term can be expanded to give

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'}\right)+\frac{\partial}{\partial u}\left(\frac{\partial F}{\partial u'}\right)\frac{du}{dx}+\frac{\partial}{\partial u'}\left(\frac{\partial F}{\partial u'}\right)\frac{du'}{dx}$$

So, Eq. (2) is equivalent to

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$$F_{u'u'}\frac{d^2u}{dx^2}+F_{u'u}\frac{du}{dx}+(F_{u'x}-F_u)=0. \qquad (3)$$

This is a second-order ODE, unless $\partial^2 F/\partial u'^2 \equiv 0,$ so in general there are two arbitrary constants to satisfy the end conditions. One can show that (3) is also equivalent to

$$\frac{1}{u'}\left[\frac{d}{dx}\left(F-\frac{\partial F}{\partial u'}\frac{du}{dx}\right)-\frac{\partial F}{\partial x}\right]=0.$$

So, if F does not involve x explicitly,

$$F-u'\frac{\partial F}{\partial u'}=C \quad \text{if} \quad \frac{\partial F}{\partial x}\equiv 0.$$

This is a *first integral* of Euler’s equation.

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Also, if F does not involve u explicitly, another first integral is

$$\frac{\partial F}{\partial u'} = C \quad \text{if} \quad \frac{\partial F}{\partial u} \equiv 0.$$

<u>Examples</u>

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Variational Notation

The notation of the calculus of variations shows many similarities to the differential calculus. Beginning with the integrand of the functional $I(u)$

$$F = F(x, u, u')$$

We substituted for

$$y = u(x)$$

with the trial function

$$\begin{aligned} y &= \bar{u}(x) = u(x) + \alpha\eta(x) \\ &= u(x) + \delta u \end{aligned}$$

where $\delta u = \alpha\eta(x)$ is defined as the *first variation* of $u(x)$.

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For a fixed *x*,

$$\Delta F = F(x, u + \alpha\eta, u' + \alpha\eta') - F(x, u, u').$$

Expanding via the Taylor series,

$$\begin{aligned}\Delta F &= \left[F(x,u,u') + \left(\delta u \frac{\partial F}{\partial u} + \delta u' \frac{\partial F}{\partial u'} \right) + \text{H.O.T.} \right] - F(x,u,u') \\ &= \alpha\eta \frac{\partial F}{\partial u} + \alpha\eta' \frac{\partial F}{\partial u'} + \text{H.O.T.}\end{aligned}$$

Thus, the first variation of *F* is

$$\Delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'$$

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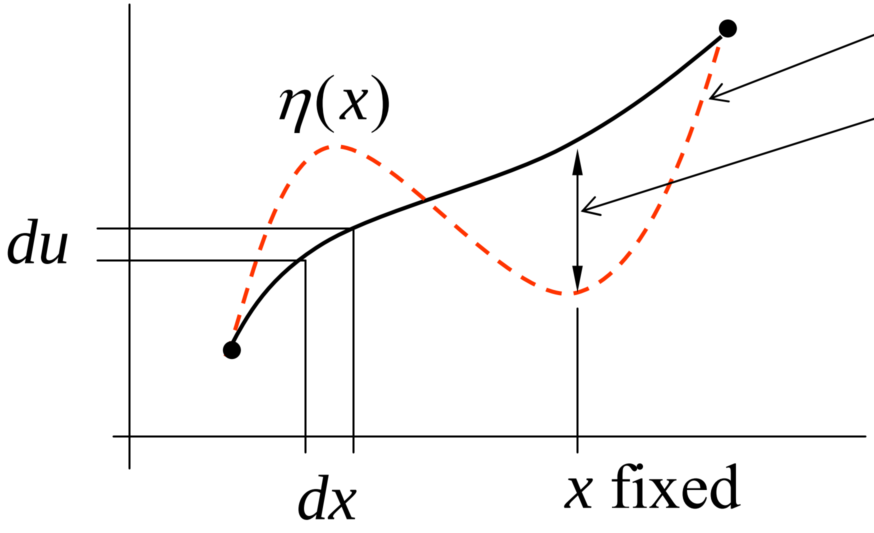
This is analogous to the total differential,

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial u'}du'$$

Graphically,



$$\bar{u}(x) = u(x) + \alpha\eta(x)$$

$$\delta u = \bar{u} - u$$

du = the first-order approximation of the change along $u(x)$ corresponding to a change in x of dx

δu = the first-order approximation to the change $\bar{u}$ to u at a fixed x

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Variational laws are analogous to differentiation,

$$\delta(F_1 F_2) = F_1 \delta F_2 + F_2 \delta F_1,$$

$$\delta\left(\frac{F_1}{F_2}\right) = \frac{F_2 \delta F_1 - F_1 \delta F_2}{F_2^2}.$$

The variation and derivative are commutative operators, i.e.,

$$\frac{d}{dx}(\delta u) = \delta\left(\frac{du}{dx}\right)$$

This will be a very important result for us later.

Applying the notation to the functional

$$I(u) = \int_a^b F(x, u, u')\,dx,$$

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Then

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\frac{d}{dx}\delta u \right] dx = 0.$$

Integrating the second term by parts,

$$\int_a^b \frac{\partial F}{\partial u'}\frac{d}{dx}\delta u \; dx = \frac{\partial F}{\partial u'}\delta u\bigg|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\delta u \; dx.$$

Then,

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]\delta u \; dx + \frac{\partial F}{\partial u'}\underset{\swarrow}{\cancel{\delta u}}\bigg|_a^b = 0$$

$\delta u = 0$ at end points

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]\delta u \; dx = 0$$

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So,

$$\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)=0.$$

Thus, we have recovered the Euler-Lagrange equation using the variational notation.

End Conditions

We now look at the end conditions (more generally, boundary conditions). Start with

$$\delta I=\delta\int_a^b F(x,u,u')dx$$

Applying Liebniz’s rule,

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$$\delta I = \int_a^b \delta F \; dx + [F]_b \cancelto{0}{\delta b} - [F]_a \cancelto{0}{\delta a} \quad \text{(fixed end points)}$$

$$= \int_a^b \left[\frac{\partial F}{\partial u} \delta u - \frac{\partial F}{\partial u'} \delta u' \right] dx.$$

Integrating by parts,

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} \right) \right] \delta u \; dx + \left[\frac{\partial F}{\partial u'} \delta u \right]_a^b$$

$$= \int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} \right) \right] \delta u \; dx + \left(\frac{\partial F}{\partial u'} \right)_b (\delta u)_b - \left(\frac{\partial F}{\partial u'} \right)_a (\delta u)_a$$

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This then gives us two possible end point conditions:

$\delta u = 0$ at $x = a, b \rightarrow$ essential boundary conditions
(u specified at $x = a$, b)

$\dfrac{\partial F}{\partial u'} = 0$ at $x = a, b \rightarrow$ natural boundary conditions
(u not specified at $x = a$, b)

Example: Calculate the first variation of $I(y) = \int_a^b y\sqrt{1+(y')^2}\;dx$

$$\delta I = \int_a^b \delta\left\{ y\sqrt{1+(y')^2} \right\} dx$$

Applying the variational chain rule:

$$\delta I = \int_a^b \left[\sqrt{1+(y')^2}\;\delta y + \frac{yy'}{\sqrt{1+(y')^2}}\,\delta y' \right] dx + \left[y\sqrt{1+(y')^2}\;\overset{= 0 \text{ (fixed endpts)}}{\cancel{\delta x}} \right]_a^b \quad \Leftarrow$$

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Higher Dimensions

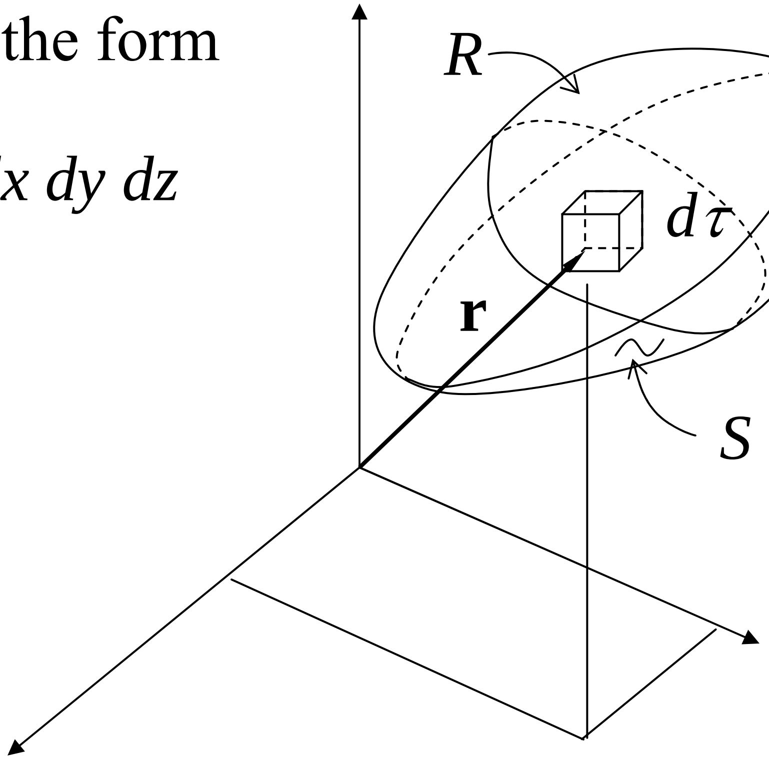
Now we extend the methods just presented to multiple variables, in particular we fist look at the function $u(x,y,z)$ and a region in space R enclosed by the surface S.

In this case, the functional $I(u)$ has the form

$$I(u) = \iiint_R F(x,y,z,u,u_x,u_y,u_z)\, dx\, dy\, dz$$

or using vector notation

$$I(u) = \iiint_R F(\mathbf{r},u,\nabla u)\, d\tau.$$



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With the variational notation, the first variation of $I(u)$ is

$$\delta I = \iiint_R \left[\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial u_z}\delta u_z \right] d\tau$$

Note that the variation is evaluated at a fixed point (x,y,z), thus δx, δy, $\delta z = 0$.

$$\delta I = \iiint_R \left[\frac{\partial F}{\partial u}\delta u + \left(\frac{\partial F}{\partial u_x}\frac{\partial}{\partial x} + \frac{\partial F}{\partial u_y}\frac{\partial}{\partial y} + \frac{\partial F}{\partial u_z}\frac{\partial}{\partial z} \right)\delta u \right] d\tau$$

Note the operator in the parentheses looks a lot like a dot product. If we set

$$\mathbf{a} = \frac{\partial F}{\partial u_x}\hat{\mathbf{e}}_x + \frac{\partial F}{\partial u_y}\hat{\mathbf{e}}_y + \frac{\partial F}{\partial u_z}\hat{\mathbf{e}}_z$$

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Then

$$\delta I = \iiint_R \left[\frac{\partial F}{\partial u} \delta u + \mathbf{a} \cdot \nabla(\delta u) \right] d\tau.$$

For integration by parts, recall

$$\nabla \cdot (\phi \mathbf{a}) = \phi \, \nabla \cdot \mathbf{a} + \nabla \phi \cdot \mathbf{a}$$

$$\mathbf{a} \cdot \nabla \phi = \nabla \cdot (\phi \mathbf{a}) - \phi \, \nabla \cdot \mathbf{a}$$

So,

$$\delta I = \iiint_R \left[\frac{\partial F}{\partial u} - \nabla \cdot \mathbf{a} \right] \delta u \, d\tau + \iiint_R \nabla \cdot (\mathbf{a} \, \delta u) \, d\tau.$$

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Now use the divergence theorem on the last integral

$$\delta I = \iiint_R \left[\frac{\partial F}{\partial u} - \nabla \cdot \mathbf{a} \right] \delta u \, d\tau + \oiint_S (\hat{\mathbf{n}} \cdot \mathbf{a}) \, \delta u \, dS.$$

Again, the necessary condition for minimizing $I(u)$ is that $\delta I = 0$. Thus, the integrand of the volume integral must be zero, i.e.,

$$\frac{\partial F}{\partial u} - \nabla \cdot \mathbf{a} = 0$$

or

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x} \right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y} \right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z} \right) = 0.$$

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This is the 3D Euler-Lagrange equation. The surface integral must also vanish, this gives three possible boundary conditions

i.) u specified on $S \rightarrow \delta u = 0$ on S (essential)

ii.) $\hat{\mathbf{n}} \cdot \mathbf{a} = \dfrac{\partial F}{\partial u_x} n_x + \dfrac{\partial F}{\partial u_y} n_y + \dfrac{\partial F}{\partial u_z} n_z = 0$ on S (natural)

iii.) $$\left.\begin{array}{l} \delta u = 0 \text{ on part of } S \\ \hat{\mathbf{n}} \cdot \mathbf{a} = \text{ on remainder of } S \end{array}\right\} \text{(mixed)}$$

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Note the steps used to obtain the Euler equation and the associated boundary conditions:

1. Take the first variation $\delta I(u)$
2. Use integration by parts to factor δu in the integrand
3. Use the divergence theorem to create a surface integral containing the boundary conditions

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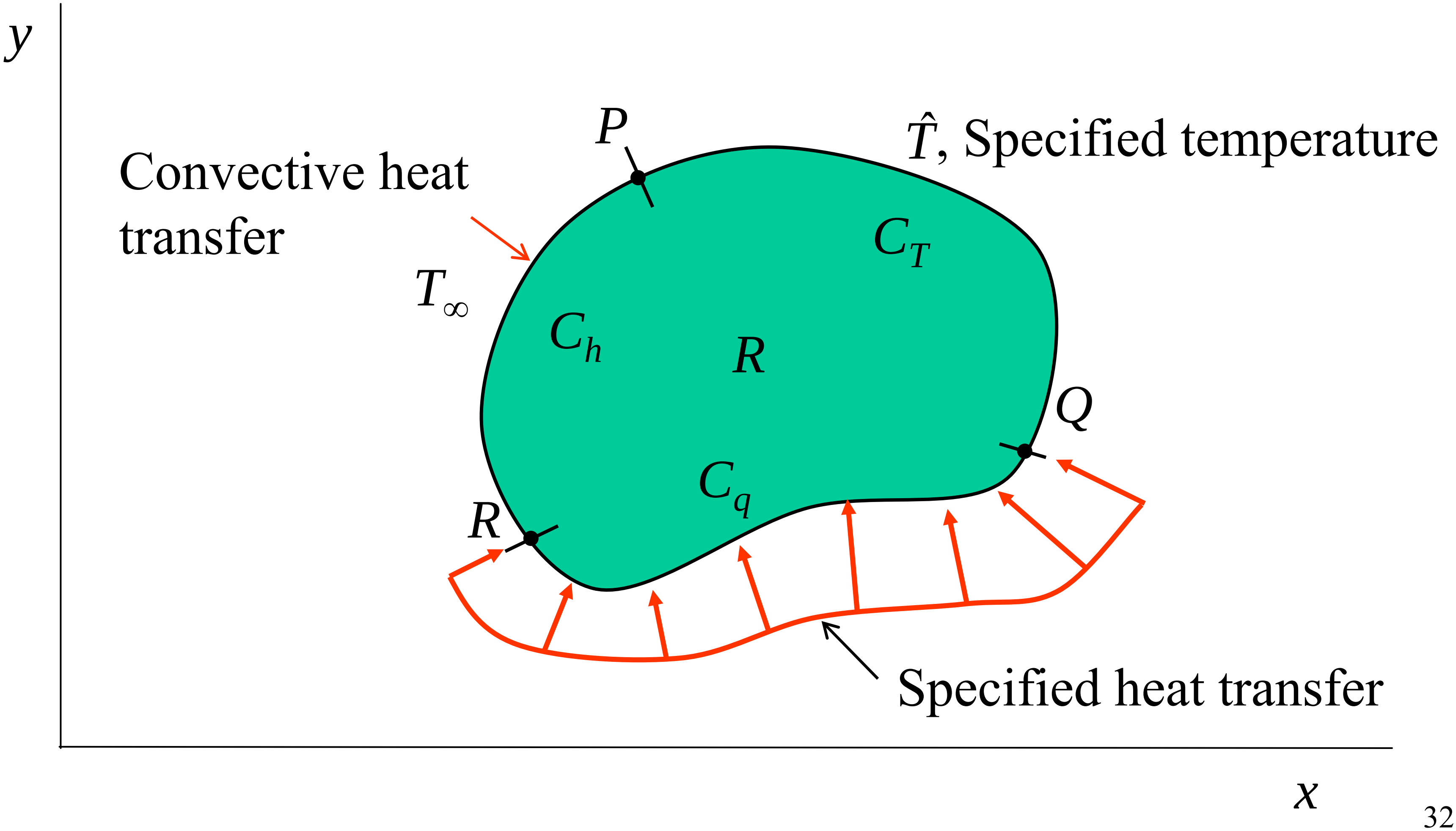
Example (from Reddy & Rasmussen): Two-dimensional conduction and convective heat transfer



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$T =$ temperature

$T_\infty =$ ambient temperature

$Q(x, y) =$ internal heat generation

$\overset{\leftrightarrow}{\mathbf{k}} = k_1\hat{\mathbf{i}}\hat{\mathbf{i}} + k_2\hat{\mathbf{j}}\hat{\mathbf{j}} =$ thermal conductivity

$\hat{q} =$ conductive heat transfer

$h =$ convective heat transfer coefficient

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Minimize:

$$I(T) = \iint_R \tfrac{1}{2}\left\{\left[\nabla T \cdot \vec{\mathbf{k}} \cdot \nabla T\right] + QT\right\} dx\,dy - \int_{C_q} \hat{q} T\, ds + \int_{C_h} h(\tfrac{1}{2} T - T_\infty) T\, ds$$

First variation:

$$\delta I(T) = \iint_R \tfrac{1}{2}\left\{\left[\nabla(\delta T) \cdot \vec{\mathbf{k}} \cdot \nabla T\right] + Q\delta T\right\} dx\,dy - \int_{C_q} \hat{q}\delta T\, ds + \int_{C_h} h(\tfrac{1}{2} T - T_\infty)\delta T\, ds$$

Note that in this problem, we already have explicit boundary integrals. As before the interior integral (over *R*) will also generate a boundary integral after it is integrated by parts.

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Integration by parts:

First, use the vector identity

$$\nabla \underbrace{\delta T}_{\phi} \cdot \underbrace{\overleftrightarrow{\mathbf{k}} \cdot \nabla T}_{\mathbf{a}} = \nabla \phi \cdot \mathbf{a},$$

$$\nabla \cdot (\phi \mathbf{a}) = \phi \nabla \cdot \mathbf{a} + \nabla \phi \cdot \mathbf{a},$$

$$\nabla \phi \cdot \mathbf{a} = \nabla \cdot (\phi \mathbf{a}) - \phi \nabla \cdot \mathbf{a}.$$

So,

$$\nabla \delta T \cdot \overleftrightarrow{\mathbf{k}} \cdot \nabla T = \nabla \cdot (\delta T \, \overleftrightarrow{\mathbf{k}} \cdot \nabla T) - \delta T \, \nabla \cdot (\overleftrightarrow{\mathbf{k}} \cdot \nabla T).$$

Now,

$$\delta I = \iint_R \left[\nabla \cdot (\delta T \, \overleftrightarrow{\mathbf{k}} \cdot \nabla T) - \delta T \, \nabla \cdot (\overleftrightarrow{\mathbf{k}} \cdot \nabla T) + Q \delta T \right] dx \, dy$$

$$- \int_{C_q} \hat{q} \delta T \, ds + \int_{C_h} h(T - T_\infty) \delta T \, ds.$$

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Now concentrate on the area integral. In 3D,

the divergence theorem gives,

$$\iiint_R \nabla\cdot(\delta T\,\overleftrightarrow{\mathbf{k}}\cdot\nabla T)\,d\tau = \oiint_S (\delta T\,\overleftrightarrow{\mathbf{k}}\cdot\nabla T)\cdot\hat{\mathbf{n}}\,dS$$

So, in 2D,

$$\iint_R \nabla\cdot(\delta T\,\overleftrightarrow{\mathbf{k}}\cdot\nabla T)\,dS = \oint_S \delta T\,\hat{\mathbf{n}}\cdot(\overleftrightarrow{\mathbf{k}}\cdot\nabla T)\,dS,$$

Then,

$$\delta I = \iint_R \left[-\delta T\,\nabla\cdot(\overleftrightarrow{\mathbf{k}}\cdot\nabla T) + Q\delta T\right]dx\,dy - \int_{C_q}\hat{q}\delta T\,ds + \int_{C_h} h(T-T_\infty)\delta T\,ds$$

$$+ \oint_{C_T+C_q+C_h} \delta T\,\hat{\mathbf{n}}\cdot(\overleftrightarrow{\mathbf{k}}\cdot\nabla T)\,dS,$$

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$$\delta I = \iint_R \left[-\nabla \cdot (\vec{\mathbf{k}} \cdot \nabla T) + Q \right] \delta T \, dx \, dy - \int_{C_T} \hat{\mathbf{n}} \cdot (\vec{\mathbf{k}} \cdot \nabla T) \delta T \, ds$$

$$+ \int_{C_q} \left[\hat{\mathbf{n}} \cdot (\vec{\mathbf{k}} \cdot \nabla T) - \hat{q} \right] \delta T \, ds + \int_{C_h} \left[\hat{\mathbf{n}} \cdot (\vec{\mathbf{k}} \cdot \nabla T) - h(T - T_\infty) \right] \delta T \, ds.$$

Now that δT is isolated we can write the Euler equation with accompanying boundary conditions

$$-\nabla \cdot (\vec{\mathbf{k}} \cdot \nabla T) + Q = 0 \qquad \text{Euler Equation}$$

essential boundary condition:

set $\int_{C_T} \hat{\mathbf{n}} \cdot (\vec{\mathbf{k}} \cdot \nabla T) \delta T \, ds = 0$ by setting $\delta T = 0$ $(T = \hat{T}$ on $C_T)$

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- Multiple Dependent Variables

So far we looked at the case where the functional is a function of one dependent variable, i.e., $I(u)$. (Note, u is a dependent variable since it depends on x.) Now we look at the case $I(u,v)$ where the functional is a function of two dependent variables.

The following development is for the 2D case, leaving the 3D case as an obvious extension. Begin with the functional to be minimized

$$I(u,v) = \iint_R F(x, y, u, v, u_x, v_x, u_y, v_y)\, dx\, dy$$

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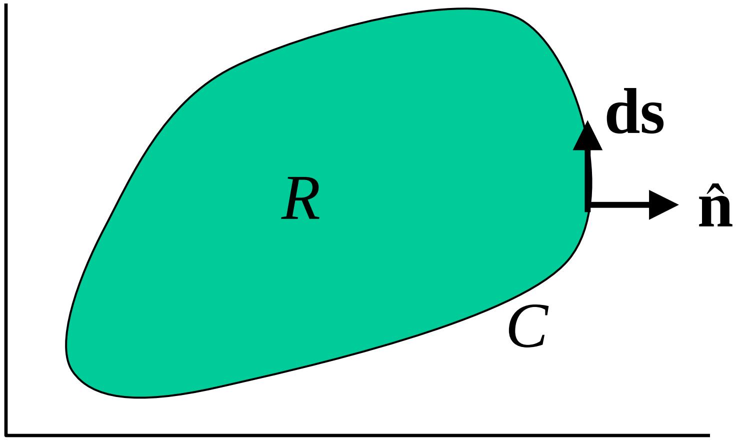
Then for the first variation

$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial v_x}\delta v_x + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial v_y}\delta v_y .$$

essential: u, v specified on $C \rightarrow \delta u = \delta v = 0$ on C.

natural: u, v not specified on C.



Calculus of Variations & Variational Methods

As before, the necessary condition for a minimum is

$$\delta I = 0 \qquad \text{for min } I(u,v)$$

Substituting the first variation δF into the integral,

$$\delta I = \iint_R \left[\left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y \right) + \left(\frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial v_x}\delta v_x + \frac{\partial F}{\partial v_y}\delta v_y \right) \right] dxdy = 0.$$

Now that we've separated the integral, it should be clear how to proceed based on our earlier developments. We next rearrange the integral to integrate by parts,

$$\delta I = \iint_R \left[\left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\frac{\partial}{\partial x}(\delta u) + \frac{\partial F}{\partial u_y}\frac{\partial}{\partial y}(\delta u) \right) + \left(\frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial v_x}\frac{\partial}{\partial x}(\delta v) + \frac{\partial F}{\partial v_y}\frac{\partial}{\partial y}(\delta v) \right) \right] dxdy = 0.$$

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Now, for integration by parts let's focus on one of the four terms in the previous expression,

$$\frac{\partial F}{\partial u_x}\frac{\partial}{\partial x}(\delta u) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u$$

Recall the gradient theorem,

$$\iiint_R \nabla\phi \, d\tau = \oiint_S \hat{\mathbf{n}}\phi \, dS.$$

Then,

$$\iint_R \nabla\phi \, dx\,dy = \oint_C \hat{\mathbf{n}}\phi \, ds \quad \Rightarrow \quad \begin{cases} \displaystyle\iint_R \frac{\partial\phi}{\partial x}\,dx\,dy = \oint_C \phi n_x \, ds \\ \displaystyle\iint_R \frac{\partial\phi}{\partial y}\,dx\,dy = \oint_C \phi n_y \, ds \end{cases}$$

Calculus of Variations & Variational Methods

So,

$$\iint_R \frac{\partial F}{\partial u_x}\frac{\partial}{\partial x}(\delta u)\,dx\,dy = \iint_R \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u\right]dx\,dy$$

$$= \oint_C \frac{\partial F}{\partial u_x} n_x\,\delta u\,ds - \iint_R \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u\,dx\,dy$$

Just as in the previous cases, the integration by parts contributes one portion to the interior of the region and one portion to the boundary. The other three similar terms are expanded in the same fashion to give,

$$\delta I = \iint_R \left\{\left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right]\delta u + \left[\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right)\right]\delta v\right\}dx\,dy$$

$$+\oint_C \left[\left(\frac{\partial F}{\partial u_x}n_x + \frac{\partial F}{\partial u_y}n_y\right)\delta u + \left(\frac{\partial F}{\partial v_x}n_x + \frac{\partial F}{\partial v_y}n_y\right)\delta v\right]ds = 0$$

Calculus of Variations & Variational Methods

So, we have

$$\left.\begin{aligned}\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)=0\\ \frac{\partial F}{\partial v}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right)=0\end{aligned}\right\}\ \text{Euler Equations}$$

$$\delta u=\delta v=0\ \text{ on } C\ \ \text{(essential)}$$

$$\left.\begin{aligned}\frac{\partial F}{\partial u_x}n_x+\frac{\partial F}{\partial u_y}n_y=0\\ \frac{\partial F}{\partial v_x}n_x+\frac{\partial F}{\partial v_y}n_y=0\end{aligned}\right\}\ \text{on } C\ \text{(natural)}$$

or mixed boundary conditions.