AEM P5.15

$9.082\ddot{y}(t) + b\dot{y}(t) + 890\,y(t) = 0$

```
clear
clf
y=dsolve('a*D2y+b*Dy+890*y=0','y(0)=0.15','Dy(0)=0','t');
y=subs(y,'a',9.082)
%
hold on
ezplot(subs(y,'b',200),[0,1])
gtext('b=200')      % Label the curve with mouse click
ezplot(subs(y,'b',179.8),[0,1])
gtext('b=179.8')
ezplot(subs(y,'b',100),[0,1])
gtext('b=100')
title('Solution to 9.082*D2y+b*Dy+890*y')
ylabel('y(t)')
hold off
```

$y(0) = 0.15 \,,\quad \dot{y}(0) = 0$

Consider spherical dilatation where

$$\vec{v} = v(r,t)\,\hat{e}_r \quad \text{and} \quad v(r,t) = r\,a(t)$$

calculate $\nabla \cdot \vec{v}$ $\qquad (h_1, h_2, h_3) = (1, r, r\sin\theta)$

$$\nabla \cdot \vec{v} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q^1}(v_1 h_2 h_3) + \frac{\partial}{\partial q^2}(v_2 h_1 h_3) + \frac{\partial}{\partial q^3}(v_3 h_1 h_2)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r\,a(t)(r)\,r\sin\theta\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^3 a\sin\theta\right)\right] = 3a \longleftarrow$$

$$\nabla \cdot \vec{v} = \left(\hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r} \frac{\partial}{\partial \theta} + \frac{\hat{e}_\phi}{r \sin\theta} \frac{\partial}{\partial \phi} \right) \cdot r\, a(t)\, \hat{e}_r$$

$$\nabla \neq \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{\partial}{\partial \phi}$$

$$\nabla \equiv \vec{e}^{\,i} \frac{\partial}{\partial q^i}$$

Hypothetical vector field

$$\vec{f}(r, \theta, \phi) = r^2 \left(\hat{e}_r + \sin\theta\, \hat{e}_\theta + \hat{e}_\phi \right)$$

a) Compute $\nabla \times \vec{f}$

b) what is the value $(r, \theta, \phi) = (2, \pi, \frac{\pi}{2})$

$$\nabla \times \vec{f} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \partial/\partial q^1 & \partial/\partial q^2 & \partial/\partial q^3 \\ h_1 f_1 & h_2 f_2 & h_3 f_3 \end{vmatrix}$$

$$(q^1, q^2, q^3) = (r, \theta, \phi)$$

$$(h_1, h_2, h_3) = (1, r, r\sin\theta)$$

$$\nabla \times \vec{f} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\, \hat{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ r^2 & r(r^2\sin\theta) & r\sin\theta(r^2) \end{vmatrix}$$

$$= \hat{e}_r \left[\frac{\partial}{\partial\theta}(r^3\sin\theta) - \frac{\partial}{\partial\phi}(r^3\sin\theta) \right]$$

$$- r\hat{e}_\theta \left[\frac{\partial}{\partial r}(r^3\sin\theta) - \frac{\partial}{\partial\phi}(r^2) \right]$$

$$+ r\sin\theta\, \hat{e}_\phi \left[\frac{\partial}{\partial r}(r^3\sin\theta) - \frac{\partial}{\partial\theta}(r^2) \right]$$

$$= r\cot\theta\,\hat{e}_r - 3r\,\hat{e}_\theta + 3r\sin\theta\,\hat{e}_\phi$$

$$\nabla\times\vec{f}\left(2,\frac{\pi}{2},\frac{\pi}{2}\right) = \cancelto{0}{\frac{2}{\tan\left(\frac{\pi}{2}\right)}}\hat{e}_r - 6\,\hat{e}_\theta + 6\sin\frac{\pi}{2}\,\hat{e}_\phi$$

$$= -6\,\hat{e}_\theta + 6\,\hat{e}_\phi$$

Given $[T_{ik}] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$,

$$\vec{c} = \hat{i}_1 + 2\hat{i}_2 + 3\hat{i}_3$$

a) write the symmetric and anti-symmetric parts

First, write each of the following in invariant (e.g. $T_{ik} = \overleftrightarrow{T}$, trace of tensor $\overleftrightarrow{T} = tr(\overleftrightarrow{T})$, etc.), then compute using the given values

b) $T_{ik} - \frac{1}{3}\delta_{ik} T_{ll}$

c) $\left(T_{ik} - \frac{1}{3}\delta_{ik} T_{ll}\right) c_i$

d) $\left(T_{ik} - \frac{1}{3}\delta_{ik} T_{ll}\right) c_i c_k$

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a) $S_{ik} = \frac{1}{2}\left(\left[T_{ik}\right] + \left[T_{ki}\right]\right)$

$$= \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}$$

$$A_{ik} = \frac{1}{2}\left([T_{ik}] - [T_{ki}]\right)$$

$$= \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

b) $$T_{ik} - \frac{1}{3}\delta_{ik} T_{ll} = \overleftrightarrow{T} - \frac{1}{3}\overleftrightarrow{I}\, tr(\overleftrightarrow{T})$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \frac{1}{3}(15)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 2 & 3 \\ 4 & 0 & 6 \\ 7 & 8 & 4 \end{bmatrix}$$

$T_{ik}\ c_i$

c) $\left(T_{ik} - \frac{1}{3}\delta_{ik} T_{ll}\right) c_i = \cancel{\vec{c}\cdot\overleftrightarrow{T}} \neq \cancel{\overleftrightarrow{T}\cdot\vec{c}}$

$\vec{c}\,\overleftrightarrow{T}$

$c_i T_{ik} \qquad T_{ki} c_i$

$c_1 T_{1k} + c_2 T_{2k} + c_3 T_{2k}$

$\neq c_1 T_{k1} + c_2 T_{k2} + c_3 T_{k3}$

$$= \vec{c}\cdot\left[\overleftrightarrow{T} - \frac{1}{3}\, tr(\overleftrightarrow{T})\,\overleftrightarrow{I}\right]$$

$$= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 2 & 3 \\ 4 & 0 & 6 \\ 7 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 25 & 26 & 27 \end{bmatrix}$$

d) $\left(T_{ik} - \frac{1}{3}\delta_{ik} T_{ll}\right) c_i c_k$ $\cancel{\vec{c}\cdot\overleftrightarrow{T} = \overleftrightarrow{T}^T\cdot\vec{c}}$ $\cancel{\boxed{(AB)^T = B^T A^T}}$

$=$

$= \vec{c}\cdot\vec{c}\,\left(\qquad \right)$

$\underbrace{T_{ik} c_i c_k}$

$= \vec{c}\vec{c} : \left[\overleftrightarrow{T} - \frac{1}{3}\mathrm{tr}(\overleftrightarrow{T})\,\overleftrightarrow{\mathbb{I}}\right]$

Set $M_{ik} = T_{ik} - \frac{1}{3}\delta_{ik} T_{ll}$

$$= M_{ik} c_i c_k = M_{11} c_1 c_1 + M_{12} c_1 c_2 + \cdots$$

$$\cdots + M_{33} c_3 c_3$$

$$= 158$$

$$\vec{c}\,\vec{c} = \begin{bmatrix} c_1 c_1 & c_1 c_2 & c_1 c_3 \\ & & \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$$

$$\overleftrightarrow{T} \cdot \vec{c} \rightarrow T\vec{c}$$

$$\vec{c} \cdot \overleftrightarrow{T} \rightarrow \vec{c}T$$

$$\vec{c} \cdot \overleftrightarrow{T} \overset{?}{=} \overleftrightarrow{T}^{T} \cdot \vec{c}$$

$$\underbrace{c_i \cdot T_{ik}} \overset{?}{=} \underbrace{T_{ki}\, c_i}$$

$$(c_1 T_{11} + c_2 T_{21} + c_3 T_{31})\hat{e}_1$$

$$+ (c_1 T_{12} + c_2 T_{22} + c_3 T_{32})\hat{e}_2 + \cdots$$

$$(T_{11} c_1 + T_{12} c_2 + T_{13} c_3)\hat{e}_1$$

$$+ (T_{21} c_1 + T_{22} c_2 + T_{23} c_3)\hat{e}_2 + \cdots$$

$$\overleftrightarrow{T} = T_{ik}\,\hat{e}_i\,\hat{e}_k \qquad \overleftrightarrow{T}^{T} \cdot \vec{c}$$

$$\vec{c} = c_j\,\hat{e}_j$$

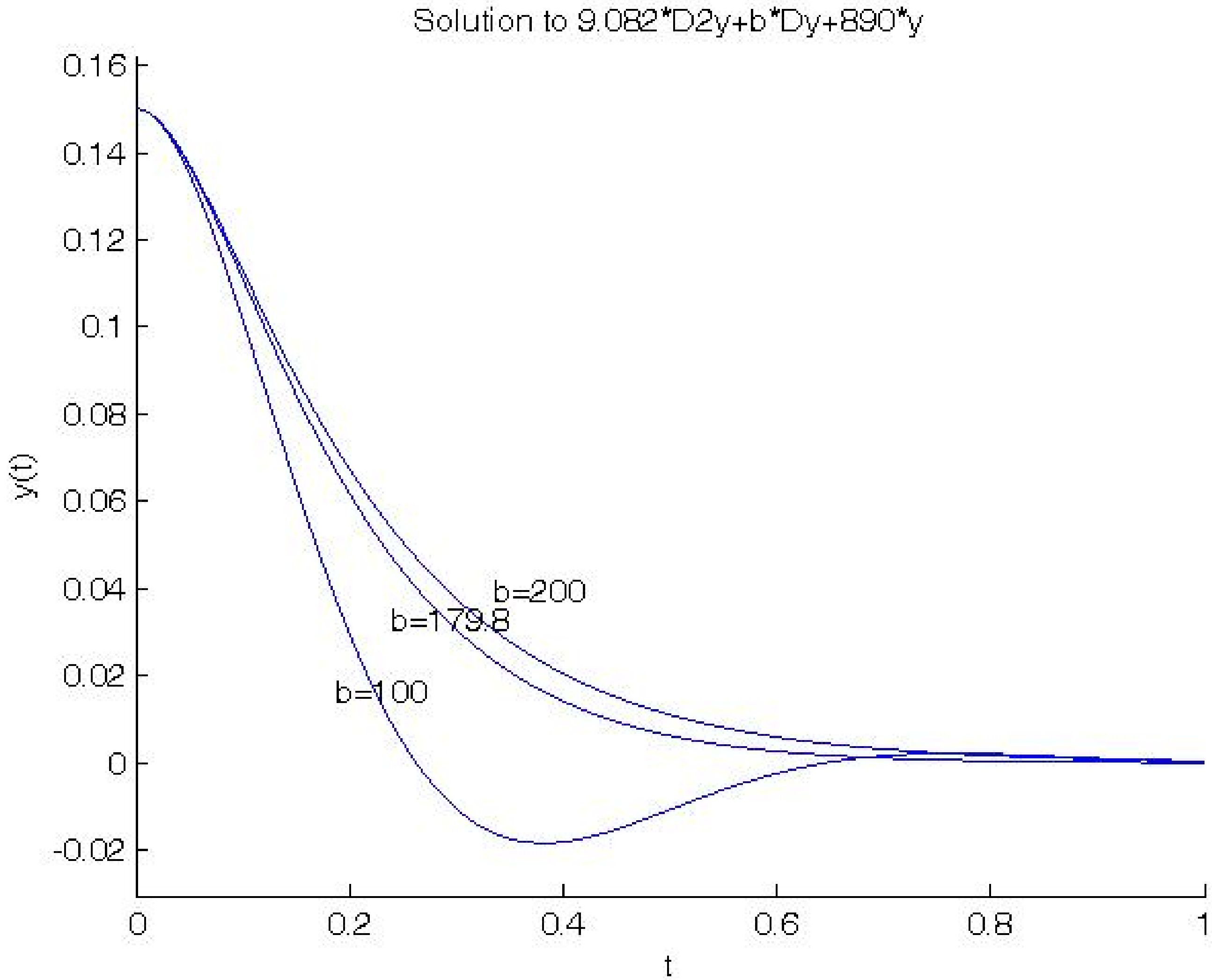
$$c_j\,\hat{e}_j \cdot T_{ik}\,\hat{e}_i\,\hat{e}_k = c_i\,T_{ik}$$

$$T_{ik}\,\hat{e}_k\,\hat{e}_i \quad \text{or} \quad T_{ki}\,\hat{e}_i\,\hat{e}_k$$

$$T_{ki}\,\hat{e}_i\,\hat{e}_k \cdot c\,\hat{e}_j = T_{ki}\,c_k$$

$$[AB]^T = B^T A^T$$

AEM P5.15


Solution to 9.082*D2y+b*Dy+890*y
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
y(t)
0
0.2
0.4
0.6
0.8
1
t
b=200
b=179.8
b=100

AEM P5.15

```
>> P5_15

y =

3/40*(b+(b^2-808298/25)^(1/2))/(b^2-808298/25)^(1/2)*exp(-
   250/4541*(b-(b^2-808298/25)^(1/2))*t)-3/40*(b-(b^2-
   808298/25)^(1/2))/(b^2-808298/25)^(1/2)*exp(-250/4541*(b+(b^2-
   808298/25)^(1/2))*t)
```

AEM P5.15

```
>> pretty(simplify(y))

       /         50
  3/40 |5 exp(- ---- (5 b - %1) t) b
       \        4541

                 50                    2          1/2
          + exp(- ---- (5 b - %1) t) (25 b  - 808298)
                 4541

                   50
          - 5 exp(- ---- (5 b + %1) t) b
                  4541

                 50                    2          1/2\   /
          + exp(- ---- (5 b + %1) t) (25 b  - 808298)   |  /
                 4541                                   / /

              2          1/2
         (25 b  - 808298)

              2          1/2
  %1 := (25 b  - 808298)
>>
```

Runge-Kutta Method: Newton's Law

AEM P5.33

Given: Newton's equation of motion for the 2-body problem

$$\ddot{x}(t) = -\frac{x(t)}{[x^2(t) + y^2(t)]^{3/2}}, \quad x(0) = 0.4, \quad \dot{x}(0) = 0,$$

$$\ddot{y}(t) = -\frac{y(t)}{[x^2(t) + y^2(t)]^{3/2}}, \quad y(0) = 0, \quad \dot{y}(0) = 2,$$

Runge-Kutta Method: Newton's Law

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= y \\ x_3 &= \dot{x} \\ x_4 &= \dot{y} \end{aligned}\right\} \rightarrow \left\{\begin{aligned} \dot{x}_1 &= \dot{x} \\ \dot{x}_2 &= \dot{y} \\ \dot{x}_3 &= \ddot{x} = -\frac{x_1}{[x_1^2 + x_2^2]^{3/2}} \\ \dot{x}_4 &= \ddot{y} = -\frac{x_2}{[x_1^2 + x_2^2]^{3/2}} \end{aligned}\right.$$

$$x_1(0) = 0.4$$

$$x_2(0) = 0$$

$$x_3(0) = 0$$

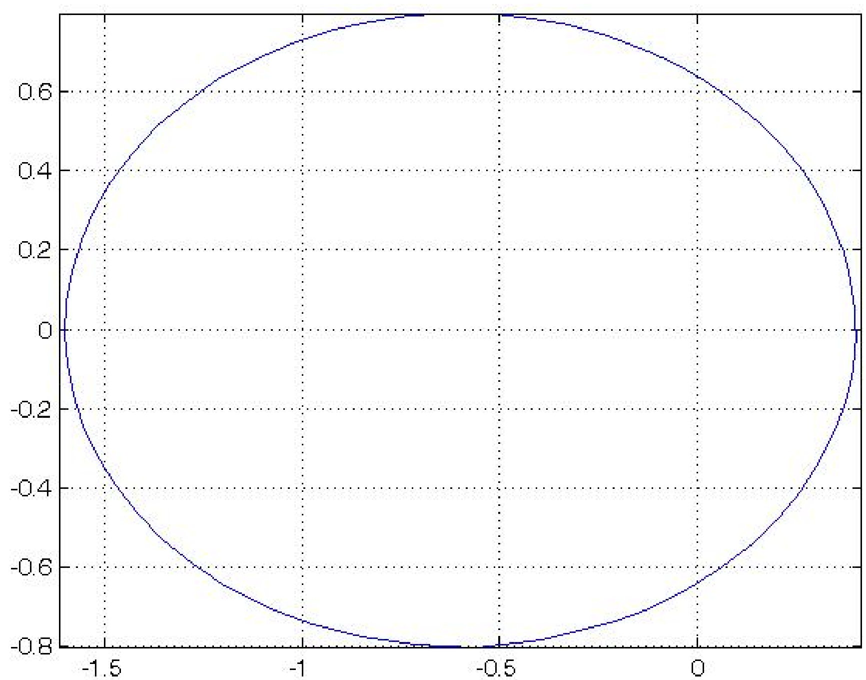
$$x_4(0) = 2$$

Runge-Kutta Method: Newton's Law

$t, \begin{bmatrix} x_1 & x_2 \\ \vdots & \vdots \end{bmatrix}$

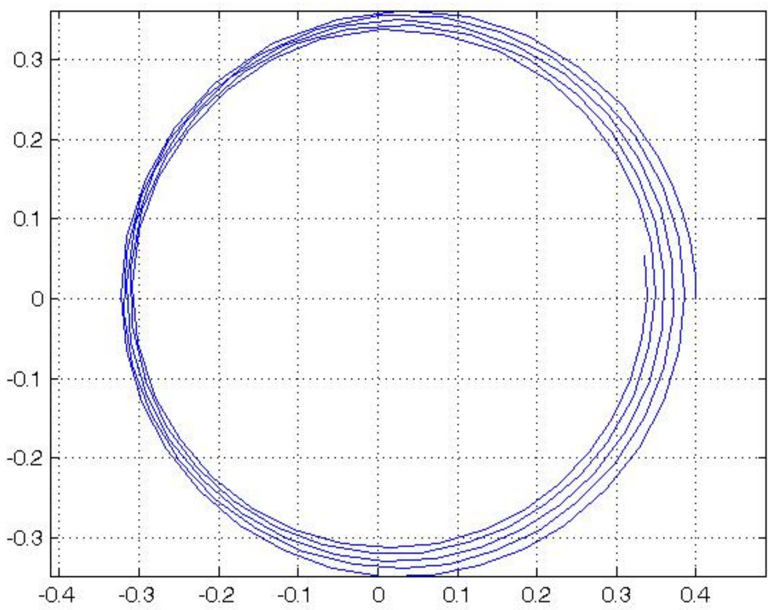
MATLAB ODE45:

```
%Script to generate orbits
%
[t,x] = ode45('orbitfun', [0 2*pi], [0.4 0 0 2]);

x1 = x(:,1);
x2 = x(:,2);

plot(x1,x2)
axis equal
grid


function dx = orbitfun(t,x)

dx = zeros(4,1);

dx(1) = x(3);
dx(2) = x(4);
dx(3) = -x(1)/(x(1)^2 + x(2)^2)^1.5;
dx(4) = -x(2)/(x(1)^2 + x(2)^2)^1.5;
```

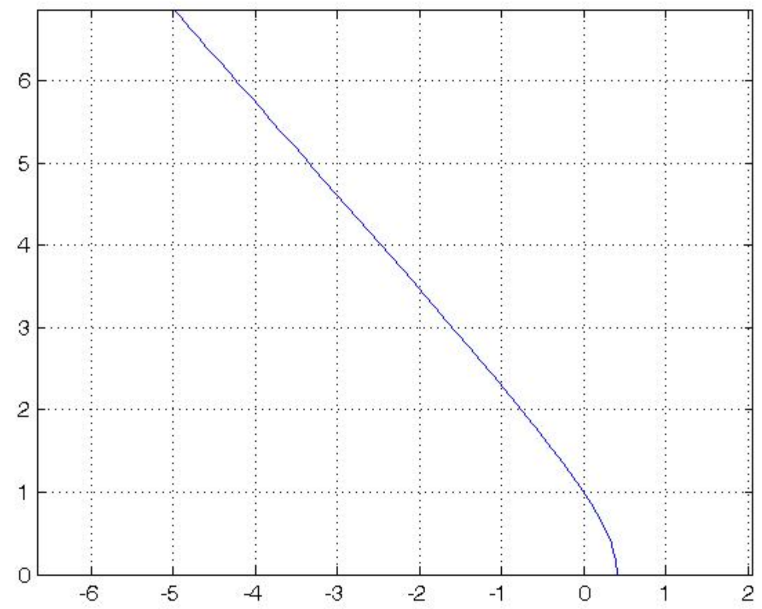


Runge-Kutta Method: Newton's Law

[t,x]=ode45('orbitfun',[0 2*pi],[0.4 0 0 1.5]);



[t,x]=ode45('orbitfun',[0 2*pi],[0.4 0 0 2.5]);



Runge-Kutta Method: Newton’s Law

```
[t,x]=ode45('orbitfun',[0 5*pi],[0.4 0 0 2.1]);
```