PART 1: VECTOR & TENSOR ANALYSIS with LINEAR ALGEBRA

Objectives

- Introduce the concepts, theories, and operational implementation of vectors, and more generally tensors, in advanced engineering analysis. The emphasis is on geometric and physical interpretations for engineering applications.

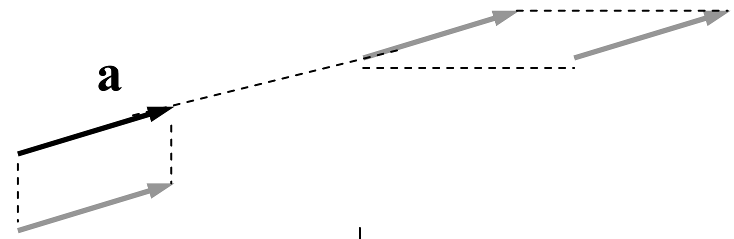
- Study some of the fundamental rules of linear algebra and show analogies with tensor analysis. We will study elementary topics of linear algebra: Matrices, determinants, systems of linear equations, and eigenvalues and eigenvectors.

Vector Definitions

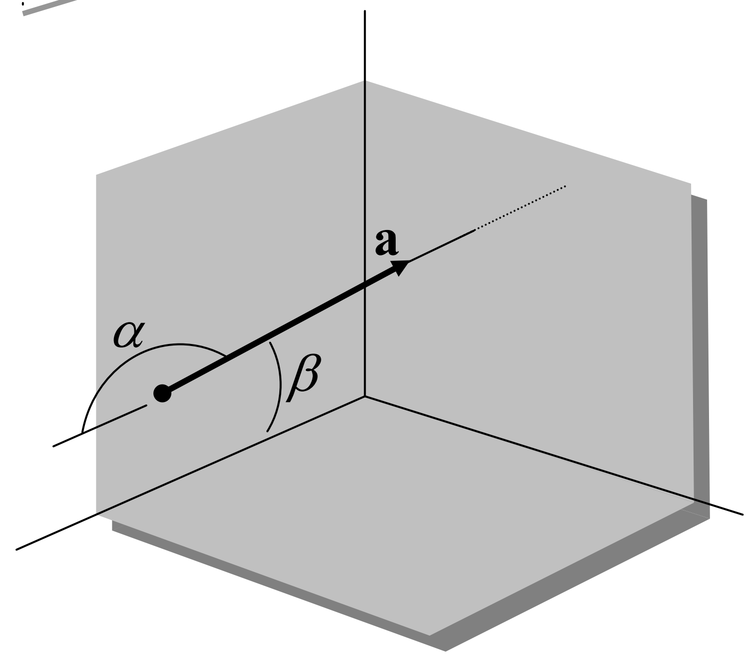
- Description of Physical Quantities
  - *Scalar:* A quantity described only by magnitude; described a single number, e.g., temperature, pressure, …
  - *Vector:* A quantity described by both magnitude and direction, e.g., velocity, displacement, …
  - *Tensor:* A higher-order vector, gives information in addition to magnitude and direction, e.g., the state of stress and strain in a continuous medium are second-order tensors.

Vector Definitions

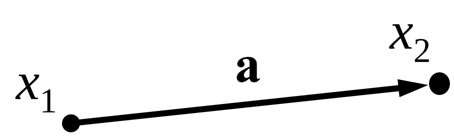
– A *free vector* can be displaced parallel to itself and act at any point; requires three numbers to specify a free vector, e.g., velocity.



– A *sliding vector* can only be displaced along a line through a fixed point containing the vector; requires five numbers to specify a sliding vector, intersection of a line and a coordinate plane (2) and the vector (3), e.g., force.

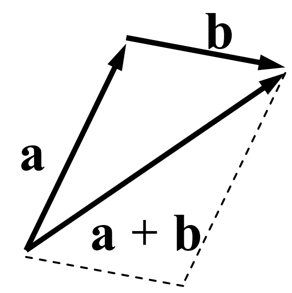


– A *bound vector* requires six numbers (coordinates points x_1, x_2), e.g., displacement.



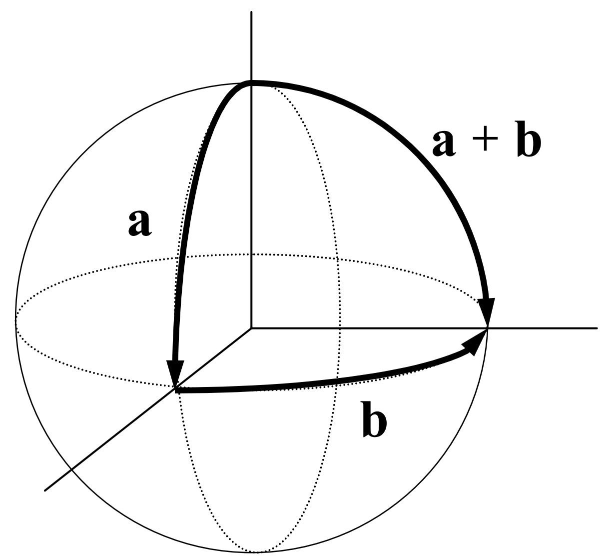
Vector Definitions

– Vectors have magnitude and direction and satisfy the parallelogram law of addition.



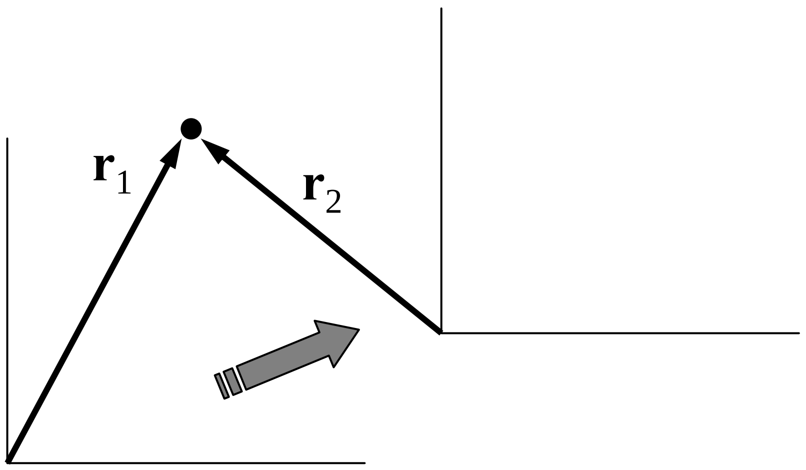
– Example: A *finite rotation* has magnitude and direction, but is it a vector . . .

. . . but the line segment (arc) that connects **a** and **b**. Therefore, a *finite rotation is not a vector since it does not satisfy the geometric definition.* (What about a differential rotation?)



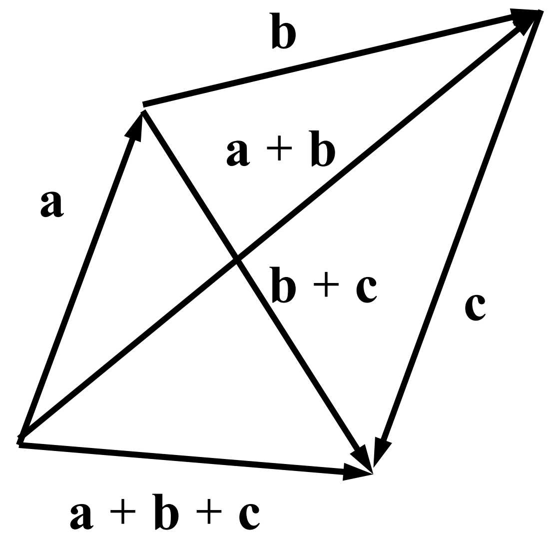
Vector Definitions

- Invariance
  - *Vectors are invariant under a coordinate transformation.*
  - Example: The position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ indicate the position of the fixed point as the coordinate system translates. After the coordinate translation, $\mathbf{r}_1 \neq \mathbf{r}_2$ therefore, *a "position vector" is really not a vector since it is not invariant under a coordinate translation!* (What about a pure coordinate rotation?)



Vector Algebra

<u>Elementary Operations</u>



– Addition

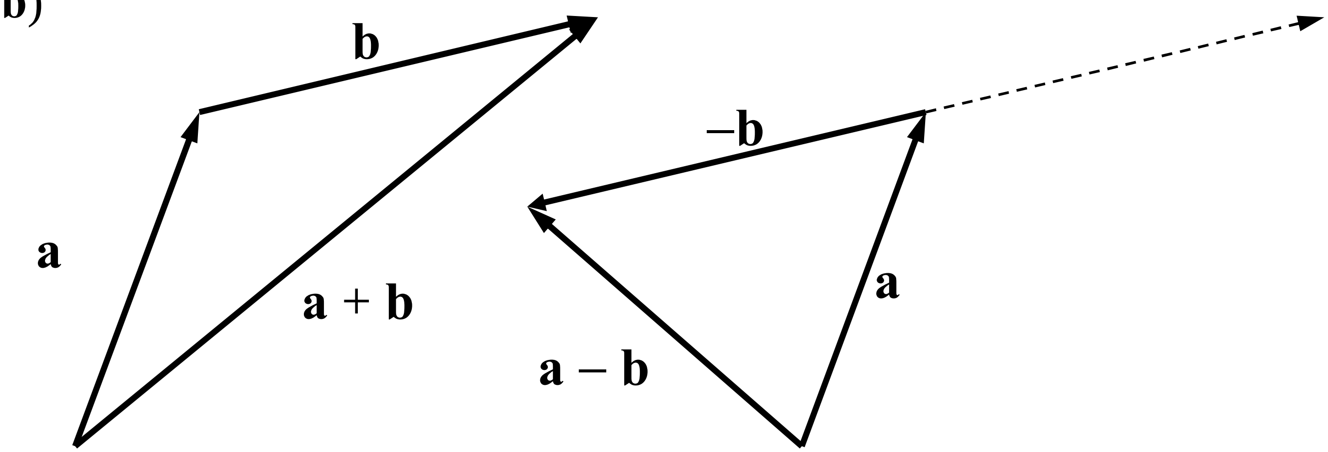
a + **b = b** + **a** → commutativity

a + **b** + **c = (a** + **b)** + **c** = **a** + **(b** + **c)** → associativity

a + **0** = **a** → additive identity

– Subtraction

a – b = a + (–**b**)



Vector Algebra

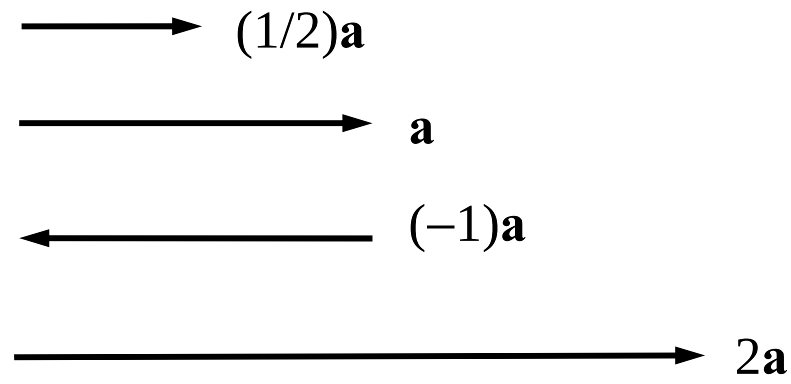
- Scalar Multiplication

  $m\mathbf{a} = \mathbf{a}m$

  $|m\mathbf{a}| = m|\mathbf{a}|,\ m > 0$

  $|m\mathbf{a}| = -\,m|\mathbf{a}|,\ m < 0$

  $0\mathbf{a} = \mathbf{0}$



- Division is not a defined vector operation
- Unit Vector

$$\hat{\mathbf{e}}_\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}}{a} \rightarrow |\hat{\mathbf{e}}_\mathbf{a}| = 1$$

A vector can always be written in terms of pure magnitude and direction using a unit vector

$$\mathbf{a} = a\,\hat{\mathbf{e}}_\mathbf{a}$$

direction

magnitude

Vector Algebra

Linear Dependence

Given vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ and scalars $\{\beta_1, \beta_2, \ldots, \beta_n\}$, not all zero. If one can write,

$$\beta_1 \mathbf{a}_1 + \beta_2 \mathbf{a}_2 + \cdots + \beta_n \mathbf{a}_n = \mathbf{0} \qquad (1)$$

then the vectors are *linearly dependent*, i.e., one is a linear combination of the others.

Example:

$$n = 2,\ \mathbf{a}_2 = -\frac{\beta_1}{\beta_2}\mathbf{a}_1 \rightarrow \textit{colinear}$$

$$n = 3,\ \mathbf{a}_3 = -\frac{1}{\beta_3}(\beta_1 \mathbf{a}_1 + \beta_2 \mathbf{a}_2) \rightarrow \textit{coplanar}$$

If (1) cannot be satisfied, the vectors are *linearly independent*.

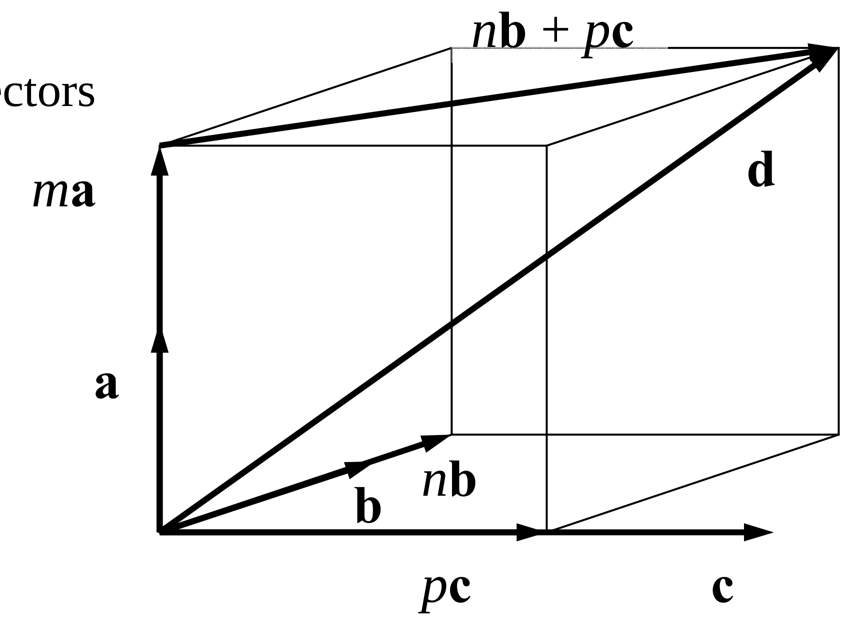
Vector Algebra

Expansion of a Vector with Respect to Other Vectors

Given **a** and **b**, linearly independent (non-colinear) then, vectors **c** and **d** can always be constructed:

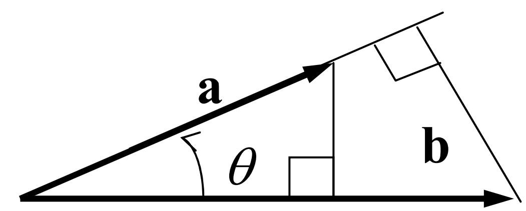
2D: $\mathbf{c} = m\mathbf{a} + n\mathbf{b}$

3D: $\mathbf{d} = m\mathbf{a} + n\mathbf{b} + p\mathbf{c}$



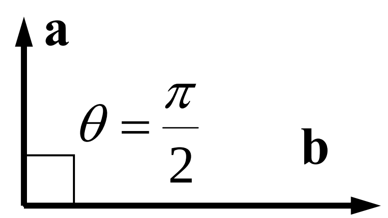
Scalar (Dot, Inner) Product

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\mathbf{a}, \mathbf{b}) = ab\cos\theta, \quad 0 \le \theta \le \pi$$

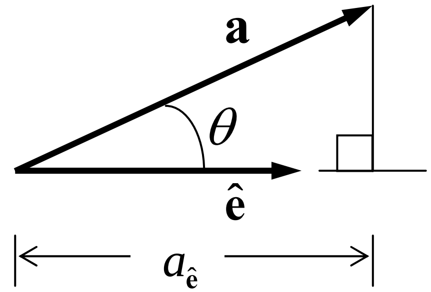


Rules

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
2. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{c})$
3. If $\mathbf{a} \perp \mathbf{b}$ (orthogonal) $\Rightarrow$ $\mathbf{a} \cdot \mathbf{b} = ab\cos(\pi/2) = 0$
4. If $\mathbf{a} \cdot \mathbf{b} = 0 \Leftrightarrow \mathbf{a} \perp \mathbf{b}$ or $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$



Vector Algebra



4. $\mathbf{a} \cdot \mathbf{a} = \mathbf{a}^2 = aa = a^2$

5. $\mathbf{a} \cdot \hat{\mathbf{e}} = a\cos\theta = a_{\mathbf{e}} \rightarrow$ projection of **a** in direction of **ê**

<u>Example:</u> Vector representation of work:

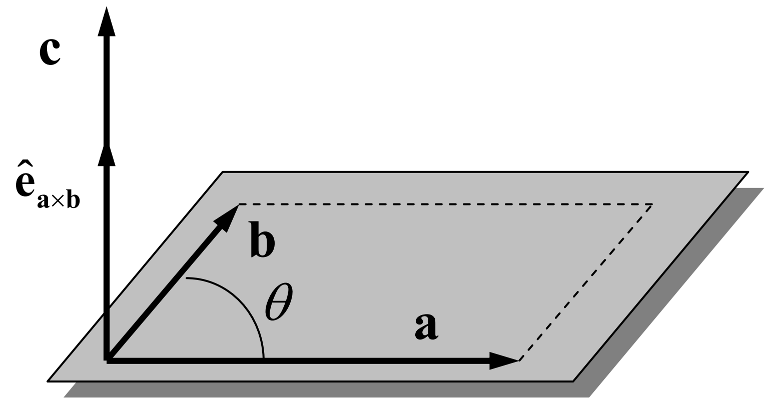
$$\text{work} = \begin{pmatrix} \text{projection of force} \\ \text{in direction} \\ \text{of displacement} \end{pmatrix} \times \begin{pmatrix} \text{magnitude of} \\ \text{displacement} \end{pmatrix}$$

$$dW = (f\cos\theta) \times (ds) = \mathbf{f} \cdot \mathbf{ds}$$

Vector (Cross, Skew, Outer) Product

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} = ab\sin\theta\hat{\mathbf{e}}_{\mathbf{a}\times\mathbf{b}}$$

The vector product obeys the *right-hand rule: Bringing* **a** *into* **b** *advances* $\hat{\mathbf{e}}_{\mathbf{a}\times\mathbf{b}}$ *in the direction of a right-handed screw.*



Vector Algebra

<u>Other rules</u>:

1. $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}\rightarrow$ anticommutativity
2. If $\mathbf{a}\parallel\mathbf{b}\Rightarrow\quad\theta=0$ or $\theta=\pi\Rightarrow\sin\theta=0\Rightarrow\mathbf{a}\times\mathbf{b}=\mathbf{0}$.

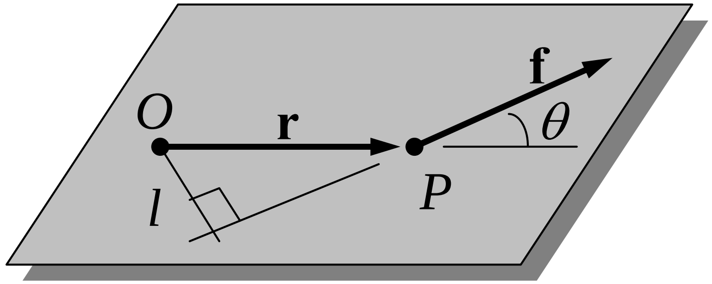
If $\mathbf{a}\times\mathbf{b}=\mathbf{0}$ then either $\mathbf{a}\parallel\mathbf{b}$ or $\mathbf{a}=\mathbf{0}$ or $\mathbf{b}=\mathbf{0}$.

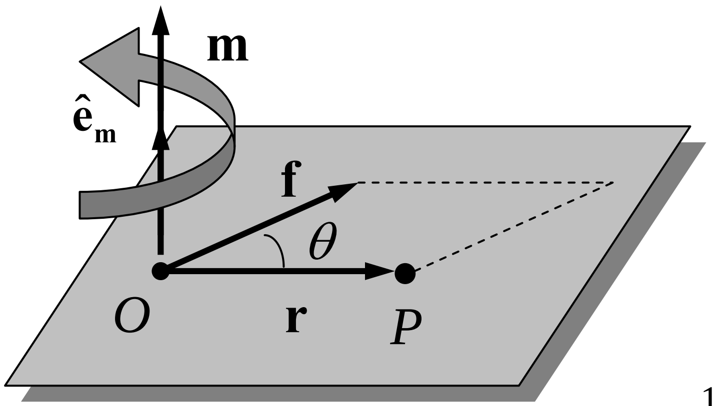
3. $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=(\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{c})\rightarrow$ distributive but order must be preserved.

<u>Example</u>: Moment (torque) about some point *O* from a force acting at a point *P*.

- $\mathbf{r}$ position of point *P* with respect to *O*
- $\mathbf{f}$ force
- $\mathbf{m}$ moment (torque)
- l perpendicular distance from *O* to line through $\mathbf{f}$

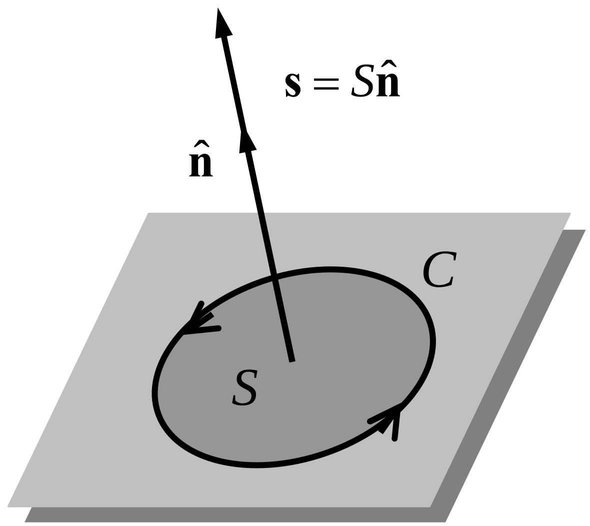
$$\mathbf{m}=\mathbf{r}\times\mathbf{f}=rf\sin\theta\,\hat{\mathbf{e}}_{\mathbf{m}}=fl\,\hat{\mathbf{e}}_{\mathbf{m}}$$





Vector Algebra

This definition of plane area can be generalized to describe a general plane area as a vector quantity. By convention, the area is enclosed on the left side when traversing the closed contour in a counterclockwise direction.



Example: Determine the projected area from the oblique cut through a circular cylinder

S = magnitude of slant area

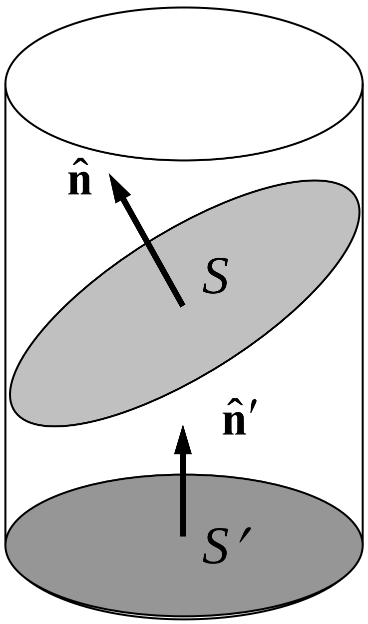
S' = magnitude of projected area

$\hat{\mathbf{n}}'$ = unit normal to area S

$\mathbf{s}' = S'\hat{\mathbf{n}}', \quad \mathbf{s} = S\hat{\mathbf{n}}$

S' is the projection of $\mathbf{s}$ in direction of $\hat{\mathbf{n}}'$

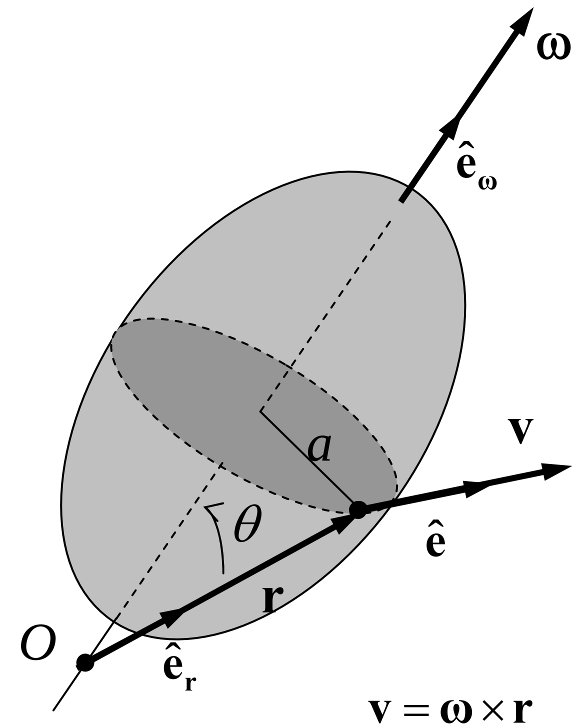
$\rightarrow \quad S' = \mathbf{s} \cdot \hat{\mathbf{n}} = S\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}'$



Vector Algebra

Rigid-Body Rotation

Determine the velocity at any point in an arbitrarily shaped, 3-D body rotating about some arbitrary axis.



r = position vector

v = linear velocity

ω = angular velocity at point *P*:

$$\mathbf{v} = \omega a \hat{\mathbf{e}}$$

from geometry:

$$a = r \sin\theta$$

$$\mathbf{v} = \omega r \sin\theta \hat{\mathbf{e}} = \boldsymbol{\omega} \times \mathbf{r}$$

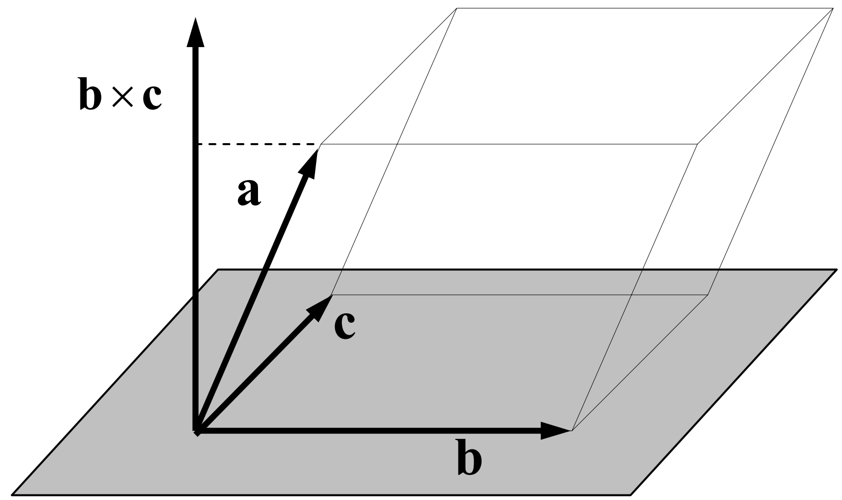
Vector Algebra

Multiple Products

<u>scalar triple product</u>: $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

1. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}\equiv[\mathbf{abc}]$
2. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{c}\cdot\mathbf{a}\times\mathbf{b}=\mathbf{b}\cdot\mathbf{c}\times\mathbf{a}$ (cyclic permutation)
3. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=-\mathbf{a}\cdot\mathbf{c}\times\mathbf{b}=-\mathbf{c}\cdot\mathbf{b}\times\mathbf{a}=-\mathbf{b}\cdot\mathbf{a}\times\mathbf{c}$ (noncyclic permutation)
4. If three vectors **a**, **b**, and **c** are coplanar, then [**abc**] = 0 (a necessary and sufficient condition.
5. [**abc**] represents the volume of a parallelapiped

volume = $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$



Vector Algebra

Multiple Products

<u>vector triple product</u>: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

1. Parentheses preserve the order of the operation and must be retained, i.e., $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$
2. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is in the plane of **b** and **c**.
3. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$

Vector Components and Basis

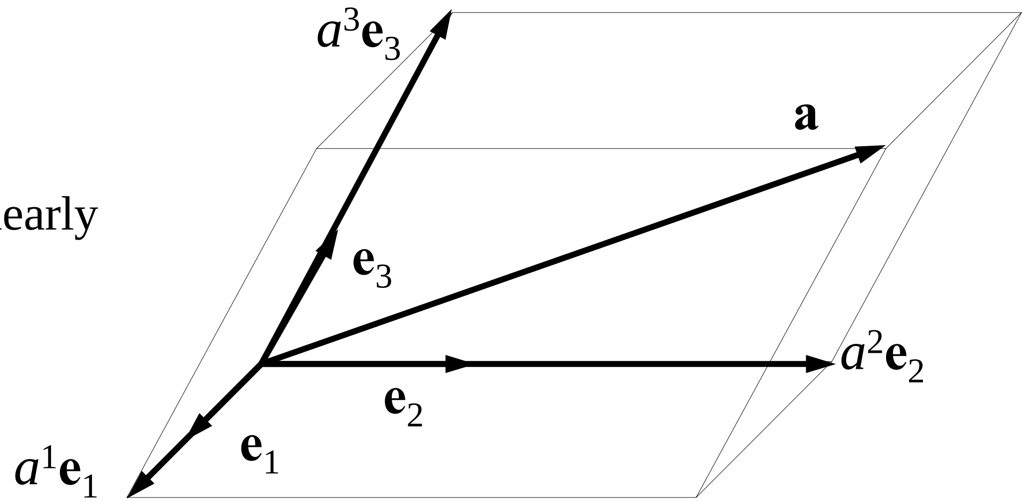
A *basis* in *n*-space contains *n* linearly independent basis vectors.

$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ represents a basis.

scalar component

$$\mathbf{a} = \boxed{a^1}\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3$$

vector component



Dual (Reciprocal) Basis

We can construct another basis $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ from $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ that enables us to obtain the scalar component of a vector.

Since $\mathbf{e}_1 \times \mathbf{e}_2$ is perpendicular to both $\mathbf{e}_1$ and $\mathbf{e}_2$, $a^3\mathbf{e}_3$ is the only nonzero component from the dot product, i.e.,

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{e}_1\times\mathbf{e}_2) &= a^1\mathbf{e}_1\cdot(\mathbf{e}_1\times\mathbf{e}_2)+a^2\mathbf{e}_2\cdot(\mathbf{e}_1\times\mathbf{e}_2)+a^3\mathbf{e}_3\cdot(\mathbf{e}_1\times\mathbf{e}_2)\\ &= a^3\mathbf{e}_3\cdot(\mathbf{e}_1\times\mathbf{e}_2)\end{aligned}$$

or

$$a^3 = \frac{\mathbf{a}\cdot(\mathbf{e}_1\times\mathbf{e}_2)}{\mathbf{e}_3\cdot(\mathbf{e}_1\times\mathbf{e}_2)} = \mathbf{a}\cdot\mathbf{e}^3,$$

where,

$$\mathbf{e}^3 = \frac{\mathbf{e}_1\times\mathbf{e}_2}{[\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3]},\ \text{and similarly, } \mathbf{e}^1 = \frac{\mathbf{e}_2\times\mathbf{e}_3}{[\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3]},\ \text{and } \mathbf{e}^2 = \frac{\mathbf{e}_3\times\mathbf{e}_1}{[\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3]}.$$

Vector Components and Basis

Now we say $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ is the dual or reciprocal basis of $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (and vice versa) since,

$$\mathbf{e}^1 \cdot \mathbf{e}_1 = \mathbf{e}^2 \cdot \mathbf{e}_2 = \mathbf{e}^3 \cdot \mathbf{e}_3 = 1.$$

Summation Convention (Einstein or Index Notation)

$$\mathbf{a} = \sum_{i=1}^{n} a^i \mathbf{e}_i \Leftrightarrow a^i \mathbf{e}_i \rightarrow \text{ sum over a repeated (dummy) index}$$

For example, in 3-space:

$$\mathbf{a} = a^1 \mathbf{e}_1 + a^2 \mathbf{e}_2 + a^3 \mathbf{e}_3$$

Kronecker Delta

With the dual basis we can now introduce a symbol called the *Kronecker delta* δ^i_j defined by,

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Since a vector is invariant to a coordinate transformation, it can be written in terms of *any* basis. In particular, we can represent an arbitrary vector **a** using the dual basis,

Vector Components and Basis

$$
\begin{aligned}
\mathbf{a} &= (\mathbf{a}\cdot\mathbf{e}_1)\mathbf{e}^1 + (\mathbf{a}\cdot\mathbf{e}_2)\mathbf{e}^2 + (\mathbf{a}\cdot\mathbf{e}_3)\mathbf{e}^3 \\
&= (a_i\mathbf{e}^i\cdot\mathbf{e}_1)\mathbf{e}^1 + (a_i\mathbf{e}^i\cdot\mathbf{e}_2)\mathbf{e}^2 + (a_i\mathbf{e}^i\cdot\mathbf{e}_3)\mathbf{e}^3 \\
&= (a_i\delta^i_1)\mathbf{e}^1 + (a_i\delta^i_2)\mathbf{e}^2 + (a_i\delta^i_3)\mathbf{e}^3 \\
&= a_1\mathbf{e}^1 + a_2\mathbf{e}^2 + a_3\mathbf{e}^3
\end{aligned}
\qquad (2)
$$

Note that the scalar components for the dual basis are written with subscripts. In general, we define,

$a_i = \mathbf{a}\cdot\mathbf{e}_i \rightarrow$ *cogredient* scalar components

$a^i = \mathbf{a}\cdot\mathbf{e}^i \rightarrow$ *contragredient* scalar components

Note that a_i transform like $\mathbf{e}_i$ and a^i transform like $\mathbf{e}^i$ since in addition to (2), we can write,

$$
\begin{aligned}
\mathbf{a} &= (\mathbf{a}\cdot\mathbf{e}^1)\mathbf{e}_1 + (\mathbf{a}\cdot\mathbf{e}^2)\mathbf{e}_2 + (\mathbf{a}\cdot\mathbf{e}^3)\mathbf{e}_3 \\
&= a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3
\end{aligned}
\qquad (3)
$$

Vector Components and Basis

For an arbitrary vector **a** written in terms of an arbitrary basis, (2) and (3) can be written as

$$\mathbf{a} = (\mathbf{a} \cdot \mathbf{e}^i)\mathbf{e}_i \text{ and } (\mathbf{a} \cdot \mathbf{e}_i)\mathbf{e}^i \qquad (4)$$

<u>Examples</u>:

Let $\mathbf{a} = a^i \mathbf{e}_i$ and $\mathbf{b} = b_j \mathbf{e}^j$, then

$$\mathbf{a} \cdot \mathbf{b} = a^i b_j (\mathbf{e}_i \cdot \mathbf{e}^j) = a^i b_j \delta_i^j = a^j b_j = a^1 b_1 + a^2 b_2 + a^3 b_3.$$

A second-order tensor might be written as,

$$\overset{\leftrightarrow}{\boldsymbol{\sigma}} = \sigma^i_j \mathbf{e}_i \mathbf{e}^j = \sigma^1_1 \mathbf{e}_1 \mathbf{e}^1 + \sigma^1_2 \mathbf{e}_1 \mathbf{e}^2 + \cdots$$

Orthonormal Basis

In general, each scalar component and basis vector has different units. For an *orthonormal* basis, the basis vectors are unit vectors (dimensionless) that are mutually perpendicular. The scalar components then have the units of the vector, i.e.,

unit + orthogonal = orthonormal

In this case the cogredient and contragredient components are the same, so

$$[\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3] = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) = \mathbf{e}_1 \cdot \mathbf{e}_1 = 1$$

and

$$\mathbf{a} = \hat{a}_1\hat{\mathbf{e}}_1 + \hat{a}_2\hat{\mathbf{e}}_2 + \hat{a}_3\hat{\mathbf{e}}_3.$$

Here, the $\hat{a}_i$ are *physical components* that have the units of the vector.

Vector Components and Basis

Most engineering applications requiring reference to a specific coordinate system employ an orthonormal system. The most commonly used are the rectangular Cartesian, cylindrical, and spherical coordinate systems. We will later examine each of these systems in considerable detail.

Gram-Schmidt Orthonormalization

Purpose: Construct an orthonormal basis from an arbitrary set of linearly independent vectors, i.e., starting with the general basis $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ we will construct the orthonormal basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \cdots, \hat{\mathbf{e}}_n\}$.

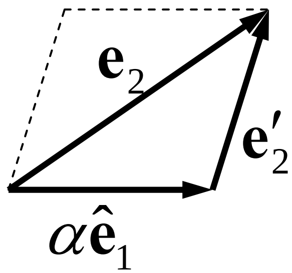
Vector Components and Basis

Why go through the trouble of creating an orthonormal basis? Because, it is generally easier to work with an orthonormal basis.

<u>Procedure</u>:

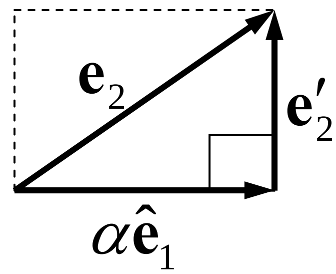
1. Given $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, normalize $\mathbf{e}_1 \rightarrow \hat{\mathbf{e}}_1 = \dfrac{\mathbf{e}_1}{|\mathbf{e}_1|}$.

2. Choose $\mathbf{e}_2$ and set $\mathbf{e}_2' = \mathbf{e}_2 - \alpha \hat{\mathbf{e}}_1$.



Vector Components and Basis

3. Require , $\mathbf{e}_2' \perp \mathbf{e}_1$



$$\hat{\mathbf{e}}_1 \cdot (\mathbf{e}_2 - \alpha\hat{\mathbf{e}}_1) = \hat{\mathbf{e}}_1 \cdot \mathbf{e}_2 - \alpha\left|\hat{\mathbf{e}}_1\right|^2 = 0 \quad \rightarrow \quad \alpha = \hat{\mathbf{e}}_1 \cdot \mathbf{e}_2$$

4. Normalize $\mathbf{e}_2' \quad \rightarrow \quad \hat{\mathbf{e}}_2 = \dfrac{\mathbf{e}_2'}{\left|\mathbf{e}_2'\right|} = \dfrac{\mathbf{e}_2 - (\hat{\mathbf{e}}_1 \cdot \mathbf{e}_2)\hat{\mathbf{e}}_1}{\left|\mathbf{e}_2'\right|}$

5. For the remaining vectors, employ the recursion relation,

$$\mathbf{e}_{r+1}' = \mathbf{e}_{r+1} - (\hat{\mathbf{e}}_1 \cdot \mathbf{e}_{r+1})\hat{\mathbf{e}}_1 - (\hat{\mathbf{e}}_2 \cdot \mathbf{e}_{r+1})\hat{\mathbf{e}}_2 - \cdots - (\hat{\mathbf{e}}_r \cdot \mathbf{e}_{r+1})\hat{\mathbf{e}}_r$$

6. Finally, normalize $\mathbf{e}_2' \quad \rightarrow \quad \hat{\mathbf{e}}_{r+1} = \dfrac{\mathbf{e}_{r+1}'}{\left|\mathbf{e}_{r+1}'\right|}$

Vector Components and Basis

Note: Graham-Schmidt orthonormalization does not necessarily yield a right-handed system. For a left-handed system, an appropriate renumbering of the orthonormal base vectors will create a right-handed system.

Permutation and Kronecker Delta Symbols

The Kronecker delta was introduced earlier where it is used in the dot product of a basis and its dual basis, i.e.,

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

For an orthonormal system, the basis and the dual basis are identical, $\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}^j$. The convention is to choose the subscript (cogredient basis) so,

Vector Components and Basis

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

The cross product operation can be represented in index notation by introducing the *permutation symbol* (actually a third-order tensor, often called the permutation tensor or alternating tensor):

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \varepsilon_{ijk} \hat{\mathbf{e}}_k \quad \text{for a right-handed orthonormal system,}$$

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{cyclic permutation of } ijk \\ -1 & \text{noncyclic permutation of } ijk \\ 0 & \text{repeated index} \end{cases}$$

Vector Components and Basis

Index Notation Examples

Applications of δ_{ij} and ε_{ijk}:

$$\mathbf{a} \cdot \mathbf{b} = (\hat{a}_i \hat{\mathbf{e}}_i) \cdot (\hat{b}_j \hat{\mathbf{e}}_j)$$

$$= \hat{a}_i \hat{b}_j \delta_{ij}$$

$$= \hat{a}_1 \hat{b}_1 \delta_{11} + \hat{a}_1 \hat{b}_2 \delta_{12} + \hat{a}_1 \hat{b}_3 \delta_{13}$$

$$+ \hat{a}_2 \hat{b}_1 \delta_{21} + \hat{a}_2 \hat{b}_2 \delta_{22} + \hat{a}_2 \hat{b}_3 \delta_{23}$$

$$+ \hat{a}_3 \hat{b}_1 \delta_{31} + \hat{a}_3 \hat{b}_2 \delta_{32} + \hat{a}_3 \hat{b}_3 \delta_{33}$$

$$= \hat{a}_1 \hat{b}_1 + \hat{a}_2 \hat{b}_2 + \hat{a}_3 \hat{b}_3$$

$$= \hat{a}_i \hat{b}_i \qquad \Leftarrow$$

Vector Components and Basis

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \hat{a}_1 & \hat{a}_2 & \hat{a}_3 \\ \hat{b}_1 & \hat{b}_2 & \hat{b}_3 \end{vmatrix}$$

$$= (\hat{a}_2\hat{b}_3 - \hat{a}_3\hat{b}_2)\hat{\mathbf{e}}_1 + (\hat{a}_3\hat{b}_1 - \hat{a}_1\hat{b}_3)\hat{\mathbf{e}}_2 + (\hat{a}_1\hat{b}_2 - \hat{a}_2\hat{b}_1)\hat{\mathbf{e}}_3$$

$$= \hat{a}_i\hat{\mathbf{e}}_i \times \hat{b}_j\hat{\mathbf{e}}_j$$

$$= \hat{a}_i\hat{b}_j\varepsilon_{ijk}\hat{\mathbf{e}}_k \quad \Leftarrow$$

The ε–δ identity: $\boxed{\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}}$

Note: Vector relations are invariant. It is convenient to develop relations and do proofs in an orthonormal (usually rectangular Cartesian) coordinate system because we can employ δ_{ij} and ε_{ijk}.

Vector Components and Basis

Example: Show that $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$

$$
\begin{aligned}
(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})& \\
&= \hat{a}_i \hat{b}_j \varepsilon_{ijk} \hat{\mathbf{e}}_k \cdot \hat{c}_m \hat{d}_n \varepsilon_{mnp} \hat{\mathbf{e}}_p \\
&= \hat{a}_i \hat{b}_j \hat{c}_m \hat{d}_n \varepsilon_{ijk} \varepsilon_{mnp} \delta_{kp} \\
&= \hat{a}_i \hat{b}_j \hat{c}_m \hat{d}_n \varepsilon_{ijk} \varepsilon_{mnk} \\
&= \hat{a}_i \hat{b}_j \hat{c}_m \hat{d}_n (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) \\
&= \hat{a}_i \hat{c}_i \hat{b}_j \hat{d}_j - \hat{a}_i \hat{d}_i \hat{b}_j \hat{c}_j \\
&= (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \quad \Leftarrow \text{ Q.E.D.}
\end{aligned}
$$

Note: An orthonormal system implies that the scalar components *are* the physical components, we will no longer employ the caret '^' above the physical components for an orthonormal system.

Basis, Dual, and Components: A Graphical Illustration

Recall from Eq. (4), $\mathbf{a} = (\mathbf{a}\cdot\mathbf{e}^i)\mathbf{e}_i$ and $\mathbf{a} = (\mathbf{a}\cdot\mathbf{e}_i)\mathbf{e}^i$. Then

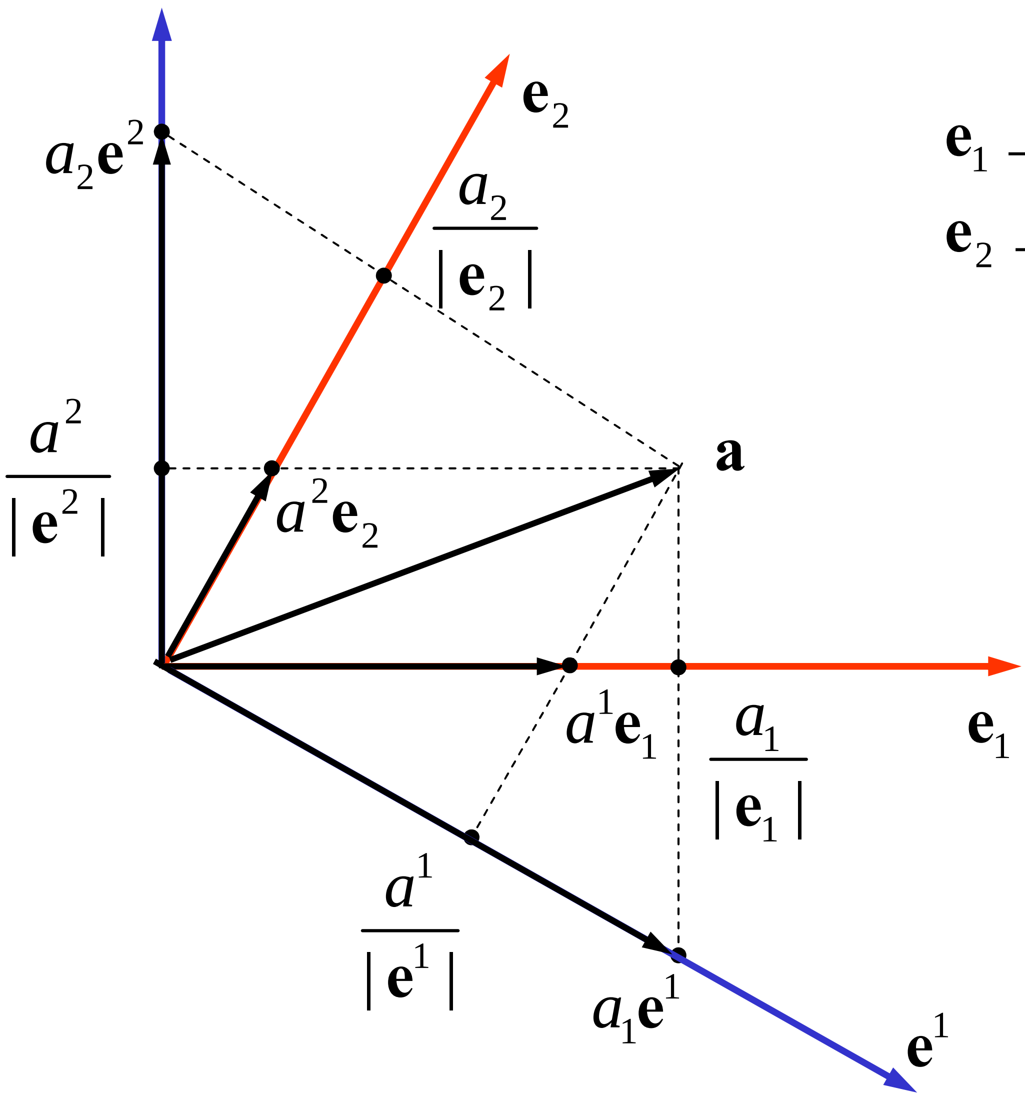
$$\mathbf{a}\cdot\mathbf{e}_i = |\mathbf{a}||\mathbf{e}_i|\cos(\mathbf{a},\mathbf{e}_i) \text{ and } (\mathbf{a}\cdot\mathbf{e}_i)\mathbf{e}^i = |\mathbf{a}||\mathbf{e}_i|\cos(\mathbf{a},\mathbf{e}^i)$$

$$\frac{a_i}{|\mathbf{e}_i|} = |\mathbf{a}|\cos(\mathbf{a},\mathbf{e}_i) \rightarrow \begin{cases}\text{orthogonal projection of } \mathbf{a} \\ \text{in the direction of } \mathbf{e}_i\end{cases}$$

$$\frac{a^i}{|\mathbf{e}^i|} = |\mathbf{a}|\cos(\mathbf{a},\mathbf{e}_i) \rightarrow \begin{cases}\text{orthogonal projection of } \mathbf{a} \\ \text{in the direction of } \mathbf{e}^i\end{cases}$$

Vector Components and Basis

2-D illustration for basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ and dual basis $\{\mathbf{e}^1, \mathbf{e}^2\}$:



$$\left.\begin{array}{l}\mathbf{e}_1 \perp \mathbf{e}^2 \\ \mathbf{e}_2 \perp \mathbf{e}^1\end{array}\right\} \rightarrow \mathbf{e}_i \perp \mathbf{e}^j = \delta_i^j$$

$$\begin{aligned}\mathbf{a} &= a^1\mathbf{e}_1 + a^2\mathbf{e}_2 \\ &= a_1\mathbf{e}^1 + a_2\mathbf{e}^2\end{aligned}$$

Vector Components and Basis

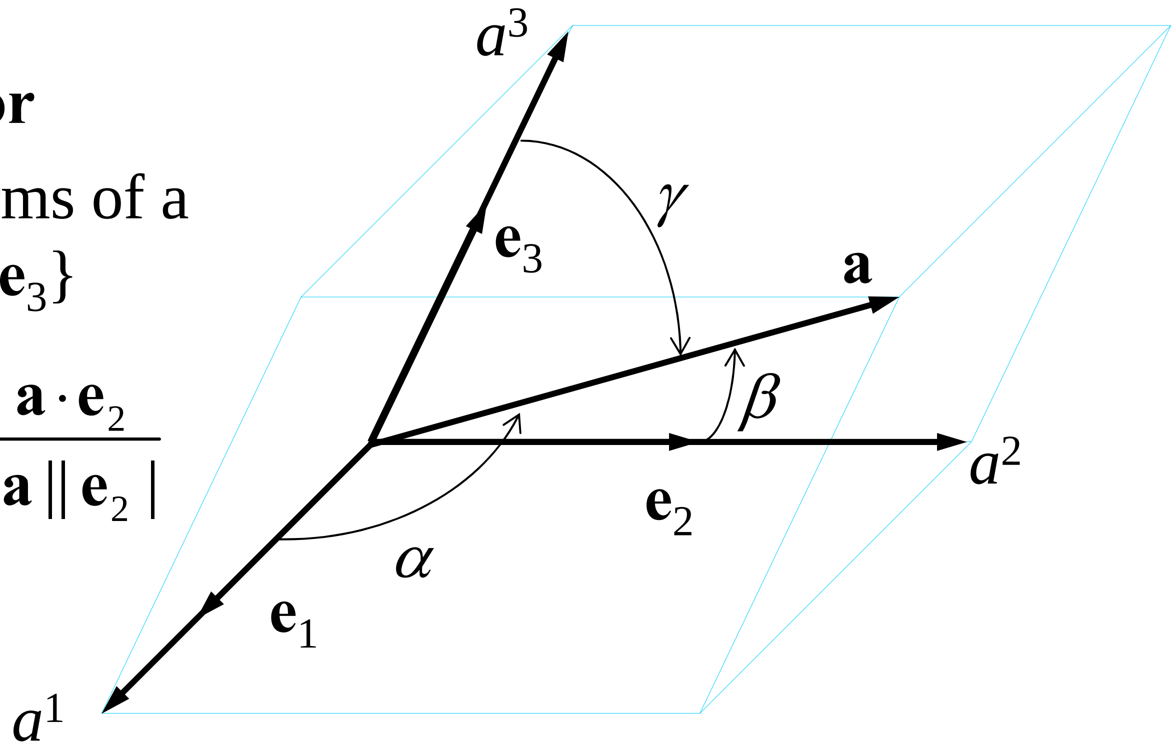
An important thing to note in the figure is that *vector* **a** *does not change in orientation or magnitude when represented in either coordinate system—it is invariant.* Note, however, that *in general, both the scalar and vector components are different for different coordinate systems.*

Specification of a Vector

Given vector **a** in terms of a general basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{e}_1}{|\mathbf{a}||\mathbf{e}_1|}, \cos\beta = \frac{\mathbf{a}\cdot\mathbf{e}_2}{|\mathbf{a}||\mathbf{e}_2|}$$

$$\cos\gamma = \frac{\mathbf{a}\cdot\mathbf{e}_3}{|\mathbf{a}||\mathbf{e}_3|}$$



Vector Components and Basis

Associated with the basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ one may write an *ordered triple*, e.g., (a^1,a^2,a^3), (a^1,β,γ), etc. The numbers are ordered in that they are placed in the order of the base vector to which they are associated and they specify the magnitude and direction of $\mathbf{a}$.

This leads to an analytical definition of a vector as: *An ordered set of numbers that obey certain specific "vector rules."* Our task now is to develop these rules.

Invariance & Transformation Laws

We stated that a vector is invariant under a coordinate transformation. This means that we may represent any vector in terms of the basis of any coordinate system.

We started with a given general covariant basis denoted as $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. We then introduced a method of constructing the dual (reciprocal) basis. Again, the purpose of the dual basis thus far is to enable a simple reciprocal scalar (dot) product operation. In engineering applications, we often have need to transform from one coordinate system to another; for example in an astrodynamics application, we might transform from a coordinate system fixed to an orbiting satellite to a geocentric (Earth-centered) system. In fluid mechanics we might transform from a body-fit coordinate system to a rectangular Cartesian computational system…

Vector Components and Basis

We now develop the general rules for such a coordinate transformation. Introduce a new cogredient basis associated with some new coordinate system, and its contragredient dual

$$\{\overline{\mathbf{e}}_1, \overline{\mathbf{e}}_2, \overline{\mathbf{e}}_3\} \qquad \{\overline{\mathbf{e}}^1, \overline{\mathbf{e}}^2, \overline{\mathbf{e}}^3\}$$

As explained earlier, we know a given vector **a** can be represented in terms of each basis and its dual, e.g.,

$$\mathbf{a} = a^i \mathbf{e}_i = a_j \mathbf{e}^j = \overline{a}^i \overline{\mathbf{e}}_i = \overline{a}_j \overline{\mathbf{e}}^j$$

Now using this relation for the contravariant components and the similar relation for the covariant components, we have for the barred system,

Vector Components and Basis

$$\bar{a}^s = (a^i \mathbf{e}_i) \cdot \bar{\mathbf{e}}^s = (\mathbf{e}_i \cdot \bar{\mathbf{e}}^s) a^i$$
$$= (a_j \mathbf{e}^j) \cdot \bar{\mathbf{e}}^s = (\mathbf{e}^j \cdot \bar{\mathbf{e}}^s) a_j$$

$$\bar{a}_s = (a^i \mathbf{e}_i) \cdot \bar{\mathbf{e}}_s = (\mathbf{e}_i \cdot \bar{\mathbf{e}}_s) a^i$$
$$= (a_j \mathbf{e}^j) \cdot \bar{\mathbf{e}}_s = (\mathbf{e}^j \cdot \bar{\mathbf{e}}_s) a_j$$

Once again use Eqs. (2) and (3),

$\mathbf{a} = (\mathbf{a} \cdot \mathbf{e}^j)\mathbf{e}_i$ and $\mathbf{a} = (\mathbf{a} \cdot \mathbf{e}_j)\mathbf{e}^j$

to write the transformation relations for the basis vectors and duals. Replacing the arbitrary vector **a** with the specific basis vectors from the barred system gives the barred basis vectors in terms of the unbarred basis vectors,

$$\bar{\mathbf{e}}^s = (\mathbf{e}_i \cdot \bar{\mathbf{e}}^s)\mathbf{e}^i$$
$$= (\mathbf{e}^i \cdot \bar{\mathbf{e}}^s)\mathbf{e}_i$$

$$\bar{\mathbf{e}}_s = (\mathbf{e}^j \cdot \bar{\mathbf{e}}_s)\mathbf{e}_j$$
$$= (\bar{\mathbf{e}}_j \cdot \bar{\mathbf{e}}_s)\mathbf{e}^j$$

Vector Components and Basis

Now define the dot products with the associated transformation laws:

Cogredient law: $\bar{\mathbf{e}}_s = a_s^j \mathbf{e}_j$ and $\bar{a}_s = a_s^j a_j$

Contragredient law: $\bar{\mathbf{e}}^s = b_j^s \mathbf{e}^j$ and $\bar{a}^s = b_j^s a^j$

Mixed laws:
$$\begin{cases} \bar{\mathbf{e}}_s = c_{js} \mathbf{e}^j \text{ and } \bar{a}_s = c_{js} a^j \\ \bar{\mathbf{e}}^s = d^{is} \mathbf{e}_i \text{ and } \bar{a}^s = d^{is} a_i \end{cases}$$

where

$$a_s^j \equiv \mathbf{e}^j \cdot \bar{\mathbf{e}}_s \qquad c_{js} \equiv \mathbf{e}_j \cdot \bar{\mathbf{e}}_s$$

$$b_i^s \equiv \bar{\mathbf{e}}^s \cdot \mathbf{e}_i \qquad d^{is} \equiv \mathbf{e}^i \cdot \bar{\mathbf{e}}^s$$

Vector Components and Basis

Be sure to recognize that the choice of letter for the dummy variables is arbitrary and were made for convenience.

If both systems are orthonormal, then

$$\hat{\bar{\mathbf{e}}}_i = (\hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j)\hat{\mathbf{e}}_j = \gamma_{ij}\hat{\mathbf{e}}_j$$

where

$$\gamma_{ij} = \cos(\hat{\bar{\mathbf{e}}}_i, \hat{\mathbf{e}}_j)$$

The a_s^j, etc. are *not* direction cosines since

$$a_s^j \equiv \mathbf{e}^j \cdot \bar{\mathbf{e}}_s \quad \text{and} \quad \cos(\mathbf{e}^j, \bar{\mathbf{e}}_s) = \frac{a_s^j}{|\mathbf{e}^j||\bar{\mathbf{e}}_s|}$$

Before continuing, we now visit the elements of linear algebra…