Given: $\phi = \phi(\mathbf{r}) = \phi(x_j)$

Show: $\text{curl}(\text{grad}\phi) \equiv 0$

$$\text{curl}(\text{grad}\phi) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \times \frac{\partial \phi}{\partial x_j} \hat{\mathbf{e}}_j$$

$$= \frac{\partial^2 \phi}{\partial x_i \partial x_j} \frac{\partial x_j}{\partial x_i} \hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j$$

$$= \frac{\partial^2 \phi}{\partial x_i \partial x_j} \delta_{ij} \varepsilon_{ijk} \hat{\mathbf{e}}_k$$

$$= \frac{\partial^2 \phi}{\partial x_j \partial x_j} \varepsilon_{jjk} \hat{\mathbf{e}}_k = 0 \Leftarrow$$