**ASEN 5227**
**Aerospace Math 1**
**Fall 2004**
Homework 9
Assigned 28 OCT, Due 5 NOV

1. In an incompressible velocity field, div **v** = 0, the viscous stress tensor is given by

$$\vec{\vec{\boldsymbol{\tau}}} = \mu[\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{T}}],$$

where $\mu$ is the coefficient of viscosity. Show that when $\mu$ is constant, the resultant force per unit volume on a fluid particle, div $\vec{\vec{\boldsymbol{\tau}}}$, is given by

$$\text{div}\,\vec{\vec{\boldsymbol{\tau}}} = \mu \nabla^2 \mathbf{v} = -\mu \text{ curl curl } \mathbf{v}.$$

2. Newton's second law of motion applied to a continuum states that the rate of change of momentum following a material region of fixed mass is equal to the sum of all the forces acting on the region. When the forces are divided into surface and body forces, Newton's second law reads:

$$\frac{D}{Dt}\iiint_R \rho \mathbf{v}\; d\tau = \oiint_S \hat{\mathbf{n}} \cdot \vec{\vec{\boldsymbol{\sigma}}}\, dS + \iiint_R \rho \mathbf{f}\, d\tau,$$

where $\vec{\vec{\boldsymbol{\sigma}}}$ is the surface stress tensor, **f** is the body force per unit mass, $\rho$ is the mass density, and **v** is the material velocity. Since the material particle mass $\rho\, d\tau$ is constant with respect to the material time derivative *D*/*Dt*, make use of the divergence theorem and obtain the differential form of Newton's second law of motion for a continuum:

$$\rho \frac{D\mathbf{v}}{Dt} = \text{div}\,\vec{\vec{\boldsymbol{\sigma}}} + \rho \mathbf{f}\ .$$

3. Take the scalar product with **v** of the equation for Newton's second law in Exercise 2 and obtain with the use of Exercise 3 from Homework #8 the equation of change for the kinetic energy of a material particle in a continuum:

$$\rho \frac{D}{Dt}\left(\frac{v^2}{2}\right) = \mathbf{v} \cdot \text{div}\,\vec{\vec{\boldsymbol{\sigma}}} + \rho \mathbf{v} \cdot \mathbf{f}.$$

How do you interpret the rate-of-work terms on the right-hand side?

4. Let *e* denote the thermodynamic internal energy per unit mass of a material. Then the equation of change for the total energy of a material region can be written:

$$\frac{D}{Dt}\iiint_R \rho\left(e + \frac{v^2}{2}\right) d\tau = \oiint_S \hat{\mathbf{n}} \cdot \vec{\vec{\boldsymbol{\sigma}}} \cdot \mathbf{v}\, dS + \iiint_R \rho \mathbf{f} \cdot \mathbf{v}\, d\tau - \oiint_S \mathbf{q} \cdot \hat{\mathbf{n}}\, dS.$$

The first two terms on the right-hand side describe the rate of work done on the material region by the surface stresses and the body forces. The third integral describes the net *outflow* of heat from the region, causing a decrease of energy inside the region. The heat-flux vector **q** describes the magnitude and direction of the flow of heat energy per unit time and per unit area.

By suitable operations obtain the differential form of the energy equation:

$$\rho \frac{D}{Dt}\left(e+\frac{v^2}{2}\right) = \text{div}\,(\ddot{\boldsymbol{\sigma}}\cdot\mathbf{v}) + \rho\mathbf{f}\cdot\mathbf{v} - \text{div}\,\mathbf{q}.$$

Subtract the contribution from kinetic energy and obtain

$$\rho \frac{De}{Dt} = \text{div}\,(\ddot{\boldsymbol{\sigma}}\cdot\mathbf{v}) - \mathbf{v}\cdot\text{div}\,\ddot{\boldsymbol{\sigma}} - \text{div}\,\mathbf{q}.$$

This is called the *thermodynamic form* of the energy equation for a continuum.

5. The total rate of work done by the surface stresses per unit volume is given by $\text{div}\,(\ddot{\boldsymbol{\sigma}}\cdot\mathbf{v})$. The rate of work done by the resultant of the surface stresses per unit volume is given by $\mathbf{v}\cdot\text{div}\,\ddot{\boldsymbol{\sigma}}$. The difference between these two terms yields the rate of work done be the surface stresses in deformation of the material particle, per unit volume. Show that this can be written as

$$\begin{aligned}\text{div}\,(\ddot{\boldsymbol{\sigma}}\cdot\mathbf{v}) - \mathbf{v}\cdot\text{div}\,\ddot{\boldsymbol{\sigma}} &= \ddot{\boldsymbol{\sigma}}:(\nabla\mathbf{v})^{\mathrm{T}} \\ &= \ddot{\boldsymbol{\sigma}}:\nabla\mathbf{v} \qquad (\ddot{\boldsymbol{\sigma}}\text{ symmetric}) \\ &= \frac{1}{2}\ddot{\boldsymbol{\sigma}}:[\nabla\mathbf{v}+(\nabla\mathbf{v})^{\mathrm{T}}] \qquad (\ddot{\boldsymbol{\sigma}}\text{ symmetric}).\end{aligned}$$