ASEN 5227
Aerospace Math 1
Fall 2004
Homework 8 EXTRA

This problem was not assigned, but might be useful for solving the other Homework #8 problems from Reddy and Rasmussen.

Let an arbitrary region in a continuous medium be denoted by *R* and the bounding, closed surface of this region be continuous and denoted by *S*. Let each point on the bounding surface move with the velocity $\mathbf{v}_s$. It can be shown that the time derivative of the volume integral over some continuous function $Q(\mathbf{r},t)$ is given by

$$\frac{d}{dt}\iiint_R Q(\mathbf{r},t)\,d\tau \equiv \iiint_R \frac{\partial Q}{\partial t}\,d\tau + \oiint_S Q\mathbf{v}\cdot\hat{\mathbf{n}}\,dS.$$

This expression for the differentiation of a volume integral with the variable limits is sometimes known as the three-dimensional *Leibniz rule*.

Let each element of mass in the medium move with the velocity $\mathbf{v}(\mathbf{r},t)$ and consider a special region *R* such that the bounding surface *S* is attached to a fixed set of material elements. Then each point of this surface moves itself with the material velocity, that is, $\mathbf{v}_s = \mathbf{v}$, and the region *R* thus contains a fixed total amount of mass since no mass crosses the boundary surface *S*. To distinguish the time rate of change of an integral over this material region, we replace d/dt with D/Dt and write

$$\frac{D}{Dt}\iiint_R Q(\mathbf{r},t)\,d\tau \equiv \iiint_R \frac{\partial Q}{\partial t}\,d\tau + \oiint_S Q\mathbf{v}\cdot\hat{\mathbf{n}}\,dS.$$

which holds for a material region, that is, a region of fixed total mass.

Show that the relation between the time derivative following an arbitrary region and the time derivative following a material region (fixed total mass) is

$$\frac{d}{dt}\iiint_R Q(\mathbf{r},t)\,d\tau \equiv \frac{D}{Dt}\iiint_R Q(\mathbf{r},t)\,d\tau + \oiint_S Q(\mathbf{v}_S - \mathbf{v})\cdot\hat{\mathbf{n}}\,dS.$$

The velocity difference $\mathbf{v} - \mathbf{v}_s$ is the velocity of the material measured relative to the velocity of the surface. The surface integral

$$\oiint_S Q(\mathbf{v} - \mathbf{v}_S)\cdot\hat{\mathbf{n}}\,dS.$$

thus measures the total *outflow* of the property *Q* from the region *R*.