ASEN 5227
Aerospace Math 1
Fall 2004
Homework 8
Assigned 21 OCT, Due 28 OCT

1. In the field description of a continuous variable $Q = Q(\mathbf{r}, t)$ let the field position be a function of time such that $\mathbf{r} = \mathbf{r}(t)$. Deduce that the total time derivative of Q can be written

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \frac{d\mathbf{r}}{dt} \cdot \text{grad}Q.$$

   This arbitrary total time derivative corresponds to a change in Q following a change in $\mathbf{r}$ with time. If we let $\mathbf{r}$ correspond to the position of a fixed material element, then $d\mathbf{r}/dt = \mathbf{v}$ corresponds to the velocity of the material element. To distinguish the time rates of change following a material element from other arbitrary changes, we write

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + \mathbf{v} \cdot \text{grad}Q.$$

   This is the differential time rate of change of a field variable $Q(\mathbf{r}, t)$ following a material element. It is referred to as the *material derivative*, the *substantial derivative*, or the *Eulerian derivative*.

   By means of vector identities show that the continuity equation for mass conservation can be written as

$$\frac{D\rho}{Dt} + \rho \text{ div } \mathbf{v} = 0.$$

2. The acceleration of a material element in a continuum is described by

$$\frac{D\mathbf{v}}{Dt} \equiv \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \text{grad } \mathbf{v}.$$

   Show by means of vector identities that the acceleration can also be written as

$$\frac{D\mathbf{v}}{Dt} \equiv \frac{\partial \mathbf{v}}{\partial t} + \text{grad } \left(\frac{\mathbf{v}^2}{2}\right) - \mathbf{v} \times \text{curl } \mathbf{v}.$$

   This form displays the role of the *vorticity vector* curl $\mathbf{v}$.

3. Deduce that

$$\mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} \equiv \frac{D}{Dt}\left(\frac{\mathbf{v}^2}{2}\right).$$

   Note that $v^2/2$ is the kinetic energy per unit mass of a material particle.

4. Newton's second law of motion in its elementary form $m\mathbf{a} = \mathbf{f}$ holds strictly for a point particle of fixed mass *m*. For a material region of continuously distributed mass, Newton's second law reads

$$\frac{D}{Dt}\iiint_R \rho\mathbf{v}\ d\tau = \mathbf{f}.$$

where **f** is the sum of all the forces acting on the material in *R*. Deduce that the corresponding law of motion for a region of *variable* mass is

$$\frac{d}{dt}\iiint_R \rho\mathbf{v}\ d\tau = \mathbf{f} + \oiint_S \rho\mathbf{v}(\mathbf{v}_S - \mathbf{v})\cdot\hat{\mathbf{n}}\,dS.$$

where $\mathbf{v}_S$ is the velocity of points on the surface *S* bounding the variable mass region *R*.

5. Define the mass in an arbitrary region by

$$m = \iiint_R \rho\ d\tau.$$

From conservation of mass it follows that

$$\dot{m} = \frac{d}{dt}\iiint_R \rho\ d\tau = \oiint_S \rho(\mathbf{v}_S - \mathbf{v})\cdot\hat{\mathbf{n}}\,dS.$$

Define the mass average velocity inside the arbitrary region *R* by

$$\mathbf{v}_a \equiv \frac{\iiint_R \rho\mathbf{v}\,d\tau}{\iiint_R \rho\,d\tau}$$

and the average efflux velocity by

$$\mathbf{v}_e \equiv \frac{\oiint_S \rho\mathbf{v}(\mathbf{v}_S - \mathbf{v})\cdot\hat{\mathbf{n}}\,dS}{\oiint_S \rho(\mathbf{v}_S - \mathbf{v})\cdot\hat{\mathbf{n}}\,dS}.$$

Show that the equation of motion, corresponding to Newton's second law, for a body of variable mass $m = m(t)$ can be written

$$\frac{d}{dt}(m\mathbf{v}_a) = \mathbf{f} + \dot{m}\mathbf{v}_e.$$

This can also be written as

$$m\frac{d\mathbf{v}_a}{dt} = \mathbf{f} + \dot{m}\mathbf{v}_{ea}.$$

where $\mathbf{v}_{ea} \equiv \mathbf{v}_e - \mathbf{v}_a$ is the average efflux velocity measured relative to the mass average velocity of the body. This last equation is sometimes referred to as the *rocket equation*. The relative moment flux $\dot{m}\mathbf{v}_{ea}$ plays the role of an apparent thrust force.