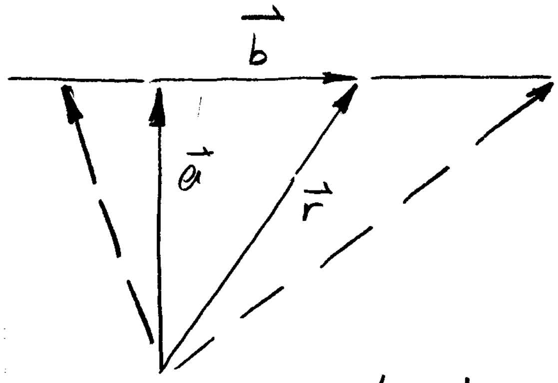


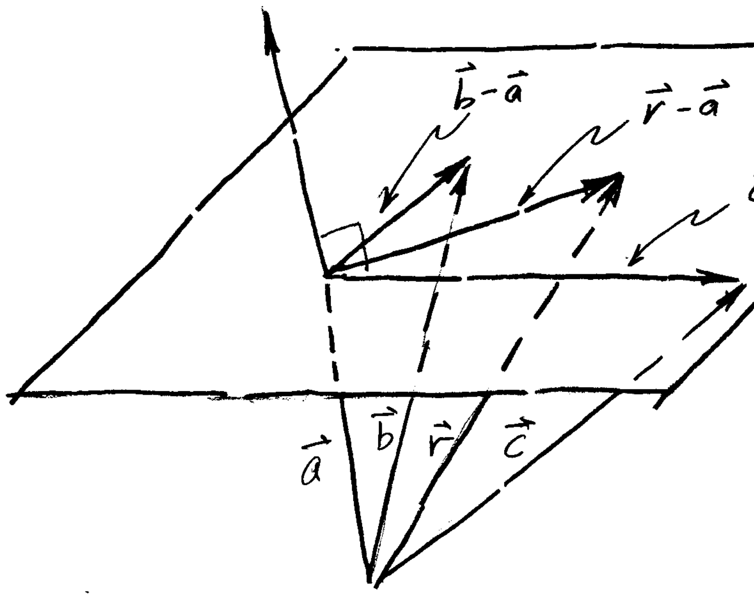
Using the parallelogram law, the line in the direction of $\vec{b}$ traced by the position arrow $\vec{r}$ can be traced by stretching $\vec{b}$:

$$\vec{r} = \vec{a} + \alpha \vec{b} \quad , \quad -\infty < \alpha < \infty$$

or

$$\vec{b} \times (\vec{r} - \vec{a}) = \vec{0}$$

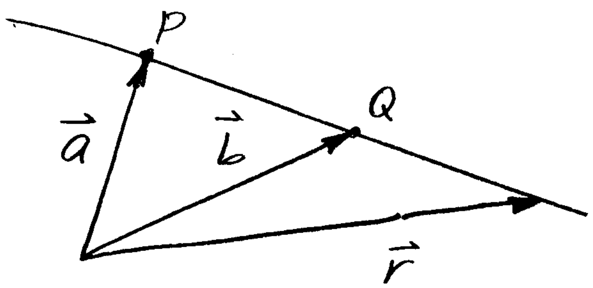
since $\vec{b}$ and $\vec{r} - \vec{a}$ are collinear.



$\vec{r}$ is a position arrow for arbitrary points on the plane

$$(\vec{c}-\vec{a})\times(\vec{b}-\vec{a}) \perp \vec{r}-\vec{a}$$

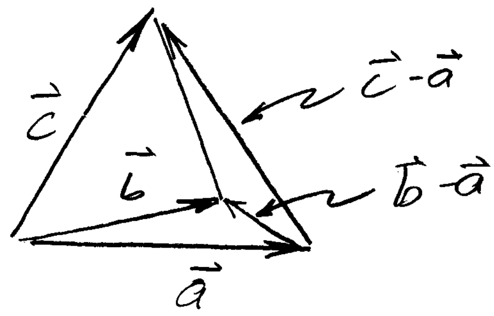
$$\rightarrow [(\vec{c}-\vec{a})\times(\vec{b}-\vec{a})]\cdot(\vec{r}\cdot\vec{a}) = 0 \leftarrow$$



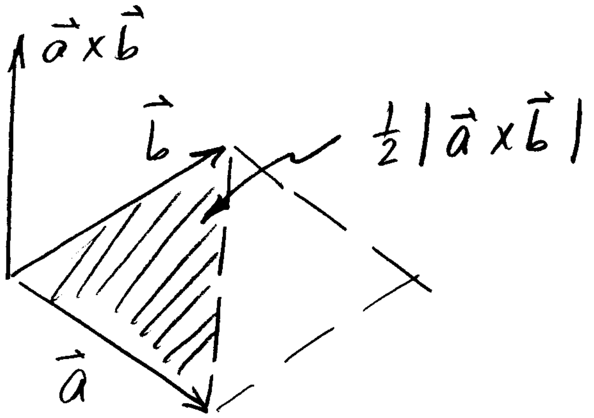
$$(\vec{r}-\vec{a}) \parallel (\vec{b}-\vec{a})$$

Using the definition of the cross product

$$(\vec{r}-\vec{a}) \times (\vec{b}-\vec{a}) = \vec{0}$$



Vector area is defined by the cross product,



$$\rightarrow \frac{1}{2}\left[\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+(\vec{c}-\vec{a})\times(\vec{b}-\vec{a})\right]$$

$$= \frac{1}{2}\left[\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{c}\times\vec{b}-\vec{c}\times\vec{a} -\vec{a}\times\vec{b}+\vec{a}\times\vec{a}\right] = \vec{0} \longleftarrow$$

QED

$$(\vec{a}\cdot\vec{b})^2 + (\vec{a}\times\vec{b})^2$$

$$= [ab\cos(\vec{a},\vec{b})]^2 + [ab\sin(\vec{a},\vec{b})\hat{e}_{\vec{a}\times\vec{b}}]^2$$

$$= (ab)^2\cos^2(\vec{a},\vec{b}) + (ab)^2\sin^2(\vec{a},\vec{b})\,|\hat{e}_{\vec{a}\times\vec{b}}|^2$$

$$= (ab)^2[\cos^2(\vec{a},\vec{b}) + \sin^2(\vec{a},\vec{b})]$$

$$= (ab)^2$$

QED

$$\vec{r}_1 = \vec{a} - 3\vec{b} + 2\vec{c}$$
$$\vec{r}_2 = 2\vec{a} - 5\vec{b} + 3\vec{c}$$
$$\vec{r}_3 = \vec{a} - 5\vec{b} + 4\vec{c}$$

Linearly independent?

if $\vec{r}_i$ are linearly dependent, then we can write

$$\beta_1 \vec{r}_1 + \beta_2 \vec{r}_2 + \beta_3 \vec{r}_3 = \vec{0}$$

$$\rightarrow \beta_1 (\vec{a} - 3\vec{b} + 2\vec{c}) + \beta_2 (2\vec{a} - 5\vec{b} + 3\vec{c}) + \beta_3 (\vec{a} - 5\vec{b} + 4\vec{c}) = \vec{0}$$

If the only solution is trivial, $\beta_1 = \beta_2 = \beta_3 = 0$ then the $\vec{r}_i$ are linearly independent. So rearrange:

$$(\beta_1 + 2\beta_2 + \beta_3)\vec{a} + (-3\beta_1 - 5\beta_2 - 5\beta_3)\vec{b} + (2\beta_1 + 3\beta_2 + 4\beta_3)\vec{c} = \vec{0}$$

Then $\vec{a} = \vec{b} = \vec{c} = \vec{0}$ is trivial, so require coefficients are separately zero,

① $\beta_1 + 2\beta_2 + \beta_3 = 0$

② $-3\beta_1 - 5\beta_2 - 5\beta_3 = 0$ $\quad \rightarrow$ eigenvalue prob

③ $2\beta_1 + 3\beta_2 + 4\beta_3 = 0$

Since this is an eigenvalue problem there are an infinite number of solutions, thus we choose

$$\beta_3 = 1 \quad \text{(arbitrary)}$$

Sub into ③

$\rightarrow 2\beta_1 + 3\beta_2 + 4 = 0 \rightarrow \beta_1 = -\frac{3}{2}\beta_2 - 2$

Sub into ① $\rightarrow -\frac{3}{2}\beta_2 - 2 + 2\beta_2 + 1 = 0$

$\frac{1}{2}\beta_2 = 1 \rightarrow \beta_2 = 2$

Sub into ① $\rightarrow \beta_1 + 4 + 1 = 0 \rightarrow \beta_1 = -5$

$\rightarrow -5\vec{r}_1 + 2\vec{r}_2 + \vec{r}_3 = \vec{0}$

$\rightarrow$ the $\vec{r}_i$ are linearly dependent $\leftarrow$