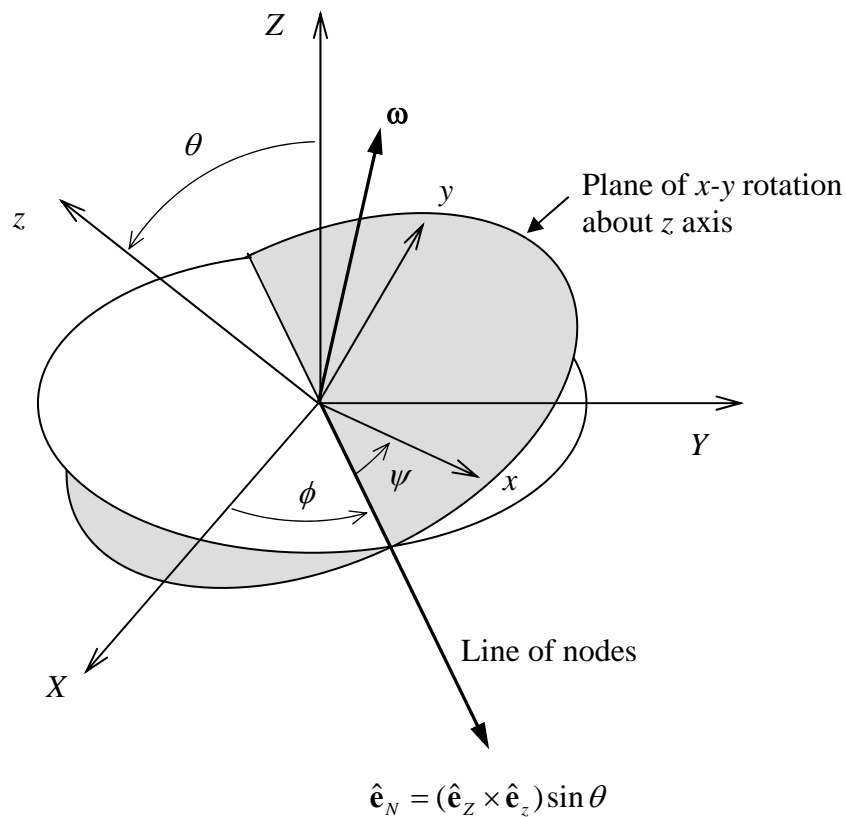


**ASEN 5227**  
**Aerospace Math 1**  
**Fall 2007**

Project 1 Supplement  
 Rotating Coordinate Frames: Euler Angles

It is useful to describe the rotational motion of rigid bodies by three angles  $\phi$ ,  $\theta$ , and  $\psi$ , called the Euler angles, as shown in the figure.<sup>1</sup> The coordinates  $X, Y, Z$  constitute an inertial rectangular Cartesian coordinate system. The  $x, y, z$  coordinates constitute a noninertial rectangular coordinate system attached to a rotating rigid body. The two coordinate systems have a common origin. The noninertial  $x, y, z$  system rotates with the angular velocity  $\omega$ . The Euler angles are delineated by three successive rotations. Let the two systems at first coincide, such that the  $x$  and  $X, y$  and  $Y, z$  and  $Z$  axes are identical. First rotate the noninertial system about the common  $z$  and  $Z$  axes by the angle  $\phi$ . Then rotate the noninertial system about its  $x$  axis, which now coincides with the line of nodes shown in the figure, by the angle  $\theta$ . Finally, rotate the noninertial system about its  $z$  axis by the angle  $\psi$ . The two angles  $\phi$  and  $\theta$  thus describe the orientation of the  $z$  axis of the noninertial system with respect to the inertial system. The angle  $\psi$  describes how much the rigid body is rotated about its  $z$  axis.



<sup>1</sup> Reddy, J.N. and Rasmussen, M.L. (1982) **Advanced Engineering Analysis**, Wiley, Chap. 1.

The rate of rotation vector  $\boldsymbol{\omega}$  can be written in terms of the three rates of change of the angles  $\phi$ ,  $\theta$ , and  $\psi$  as

$$\boldsymbol{\omega} = \dot{\phi}\hat{\mathbf{e}}_Z + \dot{\theta}\hat{\mathbf{e}}_N + \dot{\psi}\hat{\mathbf{e}}_z.$$

This description involves the unit vector  $\hat{\mathbf{e}}_Z$  pointing along the inertial  $Z$  axis, the unit vector  $\hat{\mathbf{e}}_z$  pointing along the noninertial  $z$  axis, and the unit vector  $\hat{\mathbf{e}}_N = (\hat{\mathbf{e}}_Z \times \hat{\mathbf{e}}_z) \sin \theta$  pointing along the line of nodes, which is the intersection of the plane of  $x - y$  rotation about the  $z$  axis with the  $X - Y$  plane. This representation is simple physically, but is inconvenient because the mixed basis  $\{\hat{\mathbf{e}}_Z, \hat{\mathbf{e}}_N, \hat{\mathbf{e}}_z\}$  is not orthogonal.

In applications of rigid-body dynamics it is useful to write  $\boldsymbol{\omega}$  in various different basis systems, in particular, both the inertial and the body-fixed basis systems. We thus need to establish transformations between the  $X, Y, Z$  and the  $x, y, z$  systems. We first note from geometrical considerations that

$$\begin{aligned}\hat{\mathbf{e}}_N &= \cos \phi \hat{\mathbf{e}}_X + \sin \phi \hat{\mathbf{e}}_Y \\ &= \cos \psi \hat{\mathbf{e}}_x - \sin \psi \hat{\mathbf{e}}_y.\end{aligned}$$

(a.) By similar considerations deduce that

$$\begin{aligned}\hat{\mathbf{e}}_z &= \cos \theta \hat{\mathbf{e}}_Z + \sin \theta (\hat{\mathbf{e}}_N \times \hat{\mathbf{e}}_Z) \\ \hat{\mathbf{e}}_x &= \cos \psi \hat{\mathbf{e}}_N + \sin \psi (\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_N) \\ \hat{\mathbf{e}}_y &= \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_x,\end{aligned}$$

and hence by the use of  $\hat{\mathbf{e}}_N$  that

$$\begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix} = \begin{bmatrix} \cos \phi \cos \psi - \cos \theta \sin \phi \sin \psi & \sin \phi \cos \psi + \cos \theta \cos \phi \sin \psi & \sin \theta \sin \psi \\ -\cos \phi \sin \psi - \cos \theta \sin \phi \cos \psi & -\sin \phi \sin \psi + \cos \theta \cos \phi \cos \psi & \sin \theta \cos \psi \\ \sin \theta \sin \phi & -\sin \theta \cos \phi & \cos \theta \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_X \\ \hat{\mathbf{e}}_Y \\ \hat{\mathbf{e}}_Z \end{Bmatrix}.$$

This transformation is orthogonal.

(b.) Deduce that the inverse transformation is

$$\begin{Bmatrix} \hat{\mathbf{e}}_X \\ \hat{\mathbf{e}}_Y \\ \hat{\mathbf{e}}_Z \end{Bmatrix} = \begin{bmatrix} \cos \phi \cos \psi - \cos \theta \sin \phi \sin \psi & -\cos \phi \sin \psi - \cos \theta \sin \phi \cos \psi & \sin \theta \sin \phi \\ \sin \phi \cos \psi + \cos \theta \cos \phi \sin \psi & -\sin \phi \sin \psi + \cos \theta \cos \phi \cos \psi & -\sin \theta \cos \phi \\ \sin \theta \sin \phi & \sin \theta \cos \psi & \cos \theta \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{Bmatrix}.$$

(c.) Deduce that the angular velocity can now be written in body-fixed components as

$$\boldsymbol{\omega} = (\dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi) \hat{\mathbf{e}}_x + (\dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi) \hat{\mathbf{e}}_y + (\dot{\phi} \cos \theta + \dot{\psi}) \hat{\mathbf{e}}_z$$

and in inertial components as

$$\boldsymbol{\omega} = (\dot{\psi} \sin \theta \sin \phi + \dot{\theta} \cos \phi) \hat{\mathbf{e}}_X + (-\dot{\psi} \sin \theta \cos \phi + \dot{\theta} \sin \phi) \hat{\mathbf{e}}_Y + (\dot{\psi} \cos \theta + \dot{\phi}) \hat{\mathbf{e}}_Z.$$

The set of Euler angles developed in this exercise are called the *classical Euler angle set*.<sup>2</sup> In astronomical applications, the angles are usually labeled  $(\Omega, i, \omega)$  instead of  $(\theta, \phi, \psi)$ . There are actually a total of 12 alternate combinations of 3-axis rotations that form an Euler angle set.

If we write the transformation in (b) as

$$[\hat{\mathbf{e}}] = G[\bar{\hat{\mathbf{e}}}]$$

then the Euler angles can be computed from the components of the transformation matrix  $G$  according to

$$\phi = \tan^{-1}\left(\frac{g_{13}}{-g_{23}}\right), \quad \theta = \tan^{-1}\left(\frac{\sqrt{1-g_{33}^2}}{g_{33}}\right), \quad \psi = \tan^{-1}\left(\frac{g_{31}}{g_{32}}\right)$$

Use a simple first-order backward difference to estimate the Euler angle rates, e.g.:

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{\phi(t) - \phi(t - \Delta t)}{\Delta t} + O(\Delta t).$$

Then use equations (c) to compute the angular velocity of the mothership coordinate frame. This angular velocity can then be used in the computation to transform the velocity and acceleration of the MAV, observed from the mothership (rotating frame), into the inertial coordinate frame of the ground station.

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<sup>2</sup> Henderson, D. M., **Applied Cartesian Tensors for Aerospace Simulations**, 2006, American Institute of Aeronautics and Astronautics, Chap. 1.