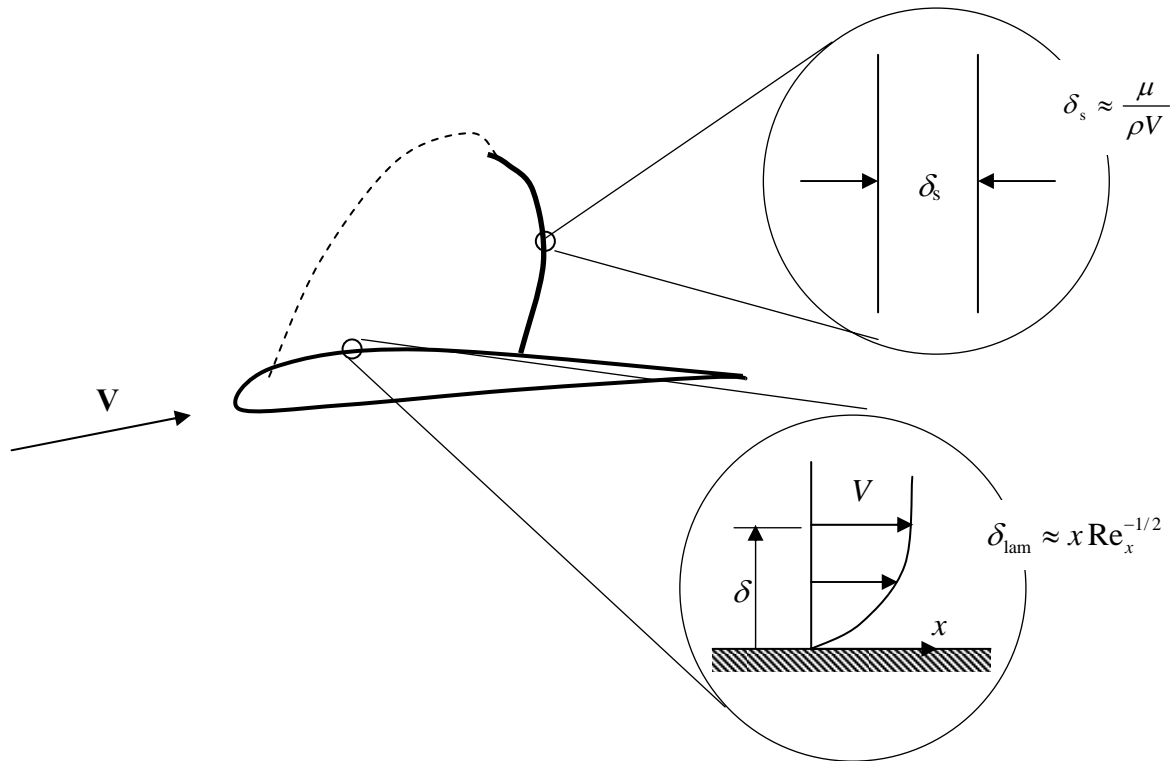


## 1 Review

The basic equations of fluid mechanics are collectively called the Navier-Stokes equations. For aerodynamic problems we know that viscous effects are confined to boundary layers and shock waves. The basic first approximation is that of an inviscid flow. That is, we consider  $Re^{-1} \rightarrow 0$ , where the Reynolds number is

$$Re = \frac{\rho V^2}{\mu V / L} = \frac{\rho V L}{\mu}$$

The sizes of the regions of viscous effects are important and are easy to estimate



The pressure field and the velocity field outside the boundary layers are determined, to lowest order, by an inviscid calculation.

### 1.1 Substantial Derivative

The equations of motion for fluid dynamics are usually described in an Eulerian system where the flow variables are described in a field. For example, the pressure  $p$  would be written as a function of spatial location and time,  $p(x,y,z,t)$ . The other system is the Lagrangian system where a fluid element is tagged and that particle is followed along its trajectory and flow quantities are described locally as the particle passes. Thus, in order to build a flow field, one would have to tag and follow the trajectories of all the fluid elements. This approach is generally not as convenient. The substantial or material derivative uses the notion of following a fluid element but retains the Eulerian field description. This operator is defined as

$$\frac{D}{Dt} \equiv \underbrace{\frac{\partial}{\partial t}}_{\text{unsteady derivative}} + \underbrace{\mathbf{V} \cdot \nabla}_{\text{convective derivative}} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}$$

The unsteady derivative accounts for time variation at a fixed point in the flow field. For instance, even for a steady flow, a fluid element will experience changes in the flow properties as it moves through the flow field. The convective derivative accounts for the variation in the flow field from a change in position.

## 1.2 Navier Stokes Equations

The conservation equations are often referred to collectively as the Navier-Stokes equations. To be correct, however, Navier-Stokes, actually only refers to the momentum equation.

Conservation of Mass:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \quad \text{or} \quad \frac{D\rho}{Dt} + \rho \frac{\partial V_i}{\partial x_i} = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho V_i) = 0.$$

Conservation of Momentum:

$$\rho \frac{D\mathbf{V}}{Dt} = \nabla \cdot \bar{\boldsymbol{\sigma}} + \rho \mathbf{f} = -\nabla p + \nabla \cdot \bar{\boldsymbol{\tau}} + \rho \mathbf{f}$$

where  $\bar{\boldsymbol{\sigma}}$  is the stress tensor,  $p$  is the hydrodynamic pressure,  $\bar{\boldsymbol{\tau}}$  is the viscous stress tensor, and  $\mathbf{f}$  is the body force per unit mass. Using Cartesian tensor notation, this is written as

$$\rho \frac{DV_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho f_i$$

The basic assumption of the Navier-Stokes equation is that the viscous stress is proportional to the viscous stress tensor is written as

$$\bar{\boldsymbol{\tau}} = \mu \left[ \nabla \mathbf{V} + (\nabla \mathbf{V})^t \right] + \lambda (\nabla \cdot \mathbf{V}) \bar{\mathbf{I}}$$

Expanding  $\bar{\boldsymbol{\tau}}$  using tensor notation,

$$\tau_{ij} = \mu \left( \frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j} \right) + \lambda \frac{\partial V_i}{\partial x_i} \delta_{ij}$$

The quantities  $\mu(\rho, T)$  and  $\lambda(\rho, T)$  are the first and second coefficients of viscosity, respectively. Note that in general, the hydrodynamic pressure and the thermodynamic pressure are not the same. For gases under the conditions of interest, we employ Stokes hypothesis

$$\mu_b = \lambda + \frac{2}{3} \mu = 0,$$

where  $\mu_b$  is the bulk viscosity. With Stokes hypothesis, the thermodynamic and hydrodynamic pressures are equal.

### Conservation of Energy:

The energy equation is an expression of the first law of thermodynamics and can be written in one of several forms

$$\rho \frac{D}{Dt} \left( e + \frac{V^2}{2} \right) = -\nabla \cdot p \mathbf{V} + \nabla \cdot (\bar{\boldsymbol{\tau}} \cdot \mathbf{V}) + \rho \mathbf{f} \cdot \mathbf{V} - \nabla \cdot \mathbf{q}$$

where  $\mathbf{q}$  is the heat flux vector described by Fourier's law:

$$\mathbf{q} = -k \nabla T$$

where  $k(\rho, T)$  is the thermal conductivity.

Although the transport coefficients have been written in general form, for gases under most engineering conditions,  $\mu(T)$ ,  $\lambda(T)$ , and  $k(T)$ .

For our purposes, a more convenient way to write the energy equation is in terms of the specific entropy  $s$

$$\rho \frac{Ds}{Dt} = \frac{\Phi}{T} - \frac{\nabla \cdot \mathbf{q}}{T} \quad \text{or} \quad \rho \frac{Ds}{Dt} = \frac{\Phi}{T} - \frac{\partial}{\partial x_i} \left( k \frac{\partial T}{\partial x_i} \right).$$

Here  $\Phi$  is the viscous dissipation function

$$\begin{aligned} \Phi &= \nabla \cdot (\bar{\boldsymbol{\tau}} \cdot \mathbf{V}) - \mathbf{V} \cdot \nabla \cdot \bar{\boldsymbol{\tau}} = \bar{\boldsymbol{\tau}} : \nabla \mathbf{V} \\ &= \tau_{ij} \frac{\partial V_j}{\partial x_i}. \end{aligned}$$

We have the basic assumptions of thermodynamic quasi-equilibrium and Kirchoff heat conduction.

### Equation of State

To close the system of equations, a state equation is required in the form

$$s = s(p, \rho) \quad \text{or} \quad p = p(s, \rho).$$

Now, we have a system of four equations with the four unknowns  $\rho$ ,  $\mathbf{V}$ ,  $s$ , and  $p$  or  $T$ . The main sources of difficulty are the nonlinearities of the system and the highest-order terms. In this

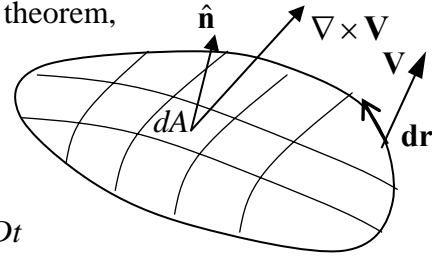
course, we are not concerned with the highest-order terms and the simplest theories neglect the nonlinear terms.

### 1.3 Some Fundamental Theorems

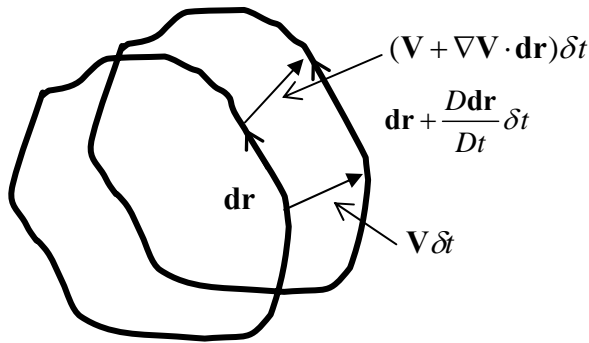
#### Kelvin's Theorem

Define  $\Gamma$ , the circulation by  $\Gamma = \oint_C \mathbf{V} \cdot d\mathbf{r}$ . Then using Stokes theorem,

$$\Gamma = \int_S (\nabla \times \mathbf{V}) \cdot \hat{\mathbf{n}} dA = \int_S \boldsymbol{\zeta} \cdot d\mathbf{A}$$



where  $\boldsymbol{\zeta} = \nabla \times \mathbf{V}$  is the vorticity. We want the result for  $D\Gamma/Dt$



$$\frac{D\Gamma}{Dt} = \oint \frac{D\mathbf{V}}{Dt} \cdot d\mathbf{r} + \oint \mathbf{V} \cdot \frac{Dd\mathbf{r}}{Dt}$$

$$\frac{Dd\mathbf{r}}{Dt} = \nabla \mathbf{V} \cdot d\mathbf{r} = dV$$

Thus, if we take  $\lambda$  and  $\mu$  to be constants and  $\mathbf{F} = \rho \mathbf{f}$

$$\frac{D\Gamma}{Dt} = \oint \left[ -\frac{\nabla p}{\rho} + \frac{\mathbf{F}}{\rho} + \frac{\lambda}{\rho} \nabla(\nabla \cdot \mathbf{V}) + \frac{\mu}{\rho} \nabla^2 \mathbf{V} + \frac{\mu}{\rho} \nabla(\nabla \cdot \mathbf{V}) \right] d\mathbf{r} + \oint \mathbf{V} \cdot dV$$

Now  $\oint dV^2 = 0$

{Note:  $\nabla^2 \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) - \nabla \times (\nabla \times \mathbf{V})$ }

Assume  $\mathbf{F}/\rho = (\text{body force})/(\text{unit mass})$  is conservative.

Then,

$$\mathbf{F} / \rho = \nabla \Omega$$

and let  $\Delta = \nabla \cdot \mathbf{V}$ ,

$$\frac{D\Gamma}{Dt} = -\oint \frac{dp}{\rho} + (\lambda + 2\mu) \oint \frac{d\Delta}{\rho} - \mu \oint \frac{\nabla \times \zeta}{\rho} \cdot d\mathbf{r}$$

1) Constant density flow

$$\nu = \mu/\rho = \text{kinematic viscosity}$$

2)  $\mu = \lambda = 0 \rightarrow$  inviscid flow

$$\frac{D\Gamma}{Dt} = -\oint \frac{dp}{\rho} = 0 \quad (\text{Bjerknes' Theorem})$$

3) If, further, the flow is barotropic (more accurately, autobarotropic) (*baros* = weight, *tropic* = turning). Then,  $p = p(\rho)$  and

$$\frac{D\Gamma}{Dt} = -\oint \frac{dp(\rho)}{\rho} = 0 \quad \underline{\text{Kelvin's Theorem}}$$

Consider the flow of a real gas with no heat addition. In regions of the flow where we can neglect the product of small coefficients with higher-order derivatives, that is viscosity and heat conduction, we may write

$$\frac{D\Gamma}{Dt} = -\oint \frac{dp}{\rho} \quad ; \quad \frac{Ds}{Dt} = 0$$

So,  $s = s(p, \rho)$  is constant following a given fluid element, and for this element then

$$\Gamma = \text{constant.}$$

If  $\Gamma = 0$  initially, then  $\Gamma = 0$  at all later times,

$$\Gamma = 0 = \oint \mathbf{V} \cdot d\mathbf{r} = \oint \zeta \cdot d\mathbf{A},$$

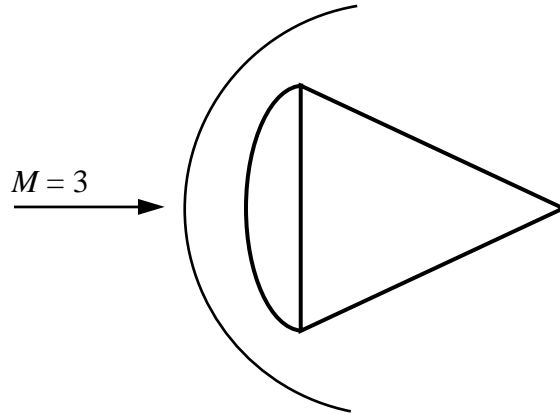
but  $d\mathbf{r}$  is arbitrary, so we find  $\zeta = \mathbf{0}$ , and we may write

$$\mathbf{V} = \nabla \phi,$$

which represents a 3:1 reduction in the number of unknowns.

The assumption that the flow is inviscid and “homentropic” will allow us to reduce our system to three scalar and one vector unknowns to a single second-order equation in one unknown.

For some important flow problems, e.g., re-entry vehicles



the failure, as we will see, encompasses the whole of the flowfield of interest. For slender bodies,



the failure is limited to the boundary layer. For a sizeable portion of this course, we will be interested in studying *adiabatic, inviscid flows without body forces*.

## 1.4 Euler Equations

Starting with the Navier Stokes equations, if we now simplify according to the assumption that  $Re \rightarrow \infty$ , we eliminate the transport terms, i.e., viscosity and heat conduction. The momentum and energy equations simplify (of course the conservation of mass equation remains unchanged). The system is collectively referred to as the Euler equations (although, again this really only refers to the momentum equations):

$$\begin{aligned}\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} &= 0 \\ \frac{D\mathbf{V}}{Dt} + \frac{\nabla p}{\rho} &= \mathbf{0} \\ T \frac{Ds}{Dt} = 0 = \frac{D\rho}{Dt} + p \frac{D}{Dt} \left( \frac{1}{\rho} \right) &= \frac{D}{Dt} \left( e + \frac{p}{\rho} \right) - \frac{1}{\rho} \frac{Dp}{Dt}\end{aligned}$$

If we now add  $\mathbf{V} \cdot \left( \frac{D\mathbf{V}}{Dt} + \frac{\nabla p}{\rho} = \mathbf{0} \right)$  to the energy equation, then

$$\frac{DH}{Dt} = \frac{1}{\rho} \frac{\partial p}{\partial t}, \quad \text{where} \quad H = e + \frac{p}{\rho} + \frac{q^2}{2} = \text{total enthalpy.}$$

Thus, if the flow is also steady, then  $H = \text{constant}$  on a particle path. We will call flows with

$$\frac{Ds}{Dt} = 0 \rightarrow s = \text{constant on a particle path, } \underline{\textit{isentropic}},$$

while if

$$s = \text{constant} = s_0 \text{ everywhere we call the flow } \underline{\textit{homentropic}}.$$

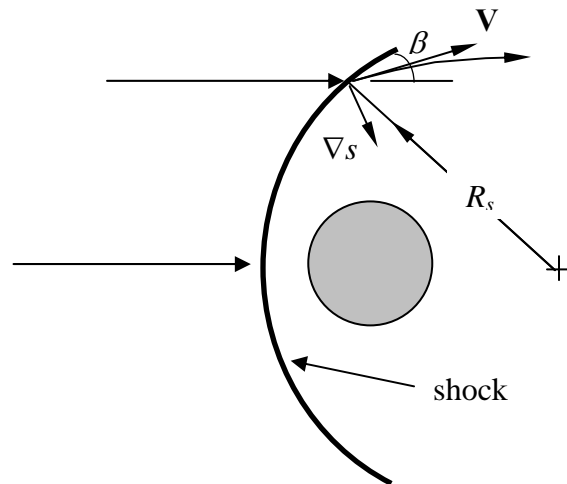
Similarly, if

$$\frac{DH}{Dt} = 0, \text{ the flow is } \underline{\textit{isoenergetic}}, \text{ and if } H = \text{constant, } \underline{\textit{homoenergetic}}.$$

For homentropic flows

$$Tds = de + pd \left( \frac{1}{\rho} \right) = 0 = dh - \frac{1}{\rho} dp.$$

Because gradients are large in both boundary layers and shock waves, we have to worry about the production of vorticity there. We can ask, "How much vorticity is introduced into the flow by shock waves?" Crocco's Theorem gives the answer for steady flow



Momentum eq.: 
$$\frac{\nabla V^2}{2} - \mathbf{V} \times \boldsymbol{\zeta} + \frac{\nabla p}{\rho} = 0$$

Energy eq.: 
$$\mathbf{dr} \cdot \left[ T \nabla_s - \nabla h - \frac{\nabla p}{\rho} \right] = 0.$$

If the flow is homogeneous,  $\mathbf{dr}$  is arbitrary and  $T \nabla_s = \nabla h - \frac{\nabla p}{\rho}$ . Thus,

$$\nabla H - T \nabla_s = \mathbf{V} \times \boldsymbol{\zeta} \quad \text{Crocco's Theorem}$$